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The Existence of Periodic Solutions of a Class of *n*-Degree Polynomial Differential Equations*

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Abstract

This paper deals with a class of *n*-degree polynomial differential equations. By the fixed point theorem and mathematical analysis techniques, the existence of one (*n* is an odd number) or two (*n* is an even number) periodic solutions of the equation is obtained. These conclusions have certain application value for judging the existence of periodic solutions of polynomial differential equations with only one higher-order term.

Keywords

n-Degree Polynomial Differential Equation, Fixed Point Theory, Periodic Solution

1. Introduction

Consider the following one element *n*-degree polynomial differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\sum_{i=0}^{n}a_i(t)x^i,\left(n\in N^+\right) \tag{1.1}$$

here, $a_i(t)(i=0,1,2,\cdots,n)$ are ω-periodic continuous real functions on R.

When $n=1$, Equation (1.1) is a linear periodic differential equation, if $\int_0^{\omega}a_1(t)\mathrm{d}t\neq 0$, then Equation (1.1) has a unique ω-periodic continuous solution (see [1]).

When $n=2$, Equation (1.1) is Riccati's equation, Riccati's equation plays an important role in fluid mechanics and the theory of elastic vibration, there are many studies on this equation [2]-[7]. In [2], the author considered the nonlinear Riccati type first-order differential equation as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=a(t)x^2+b(t), \tag{1.2}$$

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by the fixed point theory, the existence of two periodic continuous solutions of Riccati type Equation (1.2) was obtained, and the ranges of the size of the two periodic continuous solutions of Equation (1.2) were also given. One is positive, another is negative, they are symmetrical about $x=0$, we can see below for details:

Proposition 1.1 (see [2]) Consider Equation (1.2), $a(t), b(t)$ are ω-periodic continuous functions on R, suppose that the following condition holds:

$$(H_1) \quad a(t)b(t)<0, \forall t \in [0,\omega]$$

then Equation (1.2) has exactly two ω-periodic continuous solutions $\gamma_1(t)$, $\gamma_2(t)$, and

$$\sqrt{-\sup_{t\in[0,\omega]}\frac{b(t)}{a(t)}} \le \gamma_1(t) \le \sqrt{-\inf_{t\in[0,\omega]}\frac{b(t)}{a(t)}},$$

$$-\sqrt{-\inf_{t\in[0,\omega]}\frac{b(t)}{a(t)}} \le \gamma_2(t) \le -\sqrt{-\sup_{t\in[0,\omega]}\frac{b(t)}{a(t)}}.$$

When $n=3$, there are also many studies on the existence of periodic solutions of Equation (1.1) (see [8] [9] [10] [11] [12]).

So we wonder if there is a similar conclusion when n is a large positive integer?

In this paper, we are devoted to generalize Equation (1.2) to the n-th power of x and find the sufficient conditions for the existence of periodic solutions of the new equation, that is, we consider a special kind of polynomial differential Equation (1.1) as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x^n + b(t), \left(n \in N^+\right) \tag{1.3}$$

Equation (1.3) contains only an n-th power of x and a term unrelated to x, we get some similar results as Proposition 1.1 about the existence of the periodic solutions of Equation (1.3), these conclusions generalize the relevant conclusions of paper [1] and paper [2].

The rest of the paper is arranged as follows: In Section 2, some lemmas and abbreviations are introduced to be used later; In Section 3 and Section 4, the existence of periodic solutions on Equation (1.3) is obtained; In Section 5, we extend the results of Section 3 and Section 4; we end this paper with a short conclusion.

2. Preliminaries

In this section, we give some definitions, lemmas and abbreviations which will be used later.

Definition 2.1 (see [13]) Suppose $f(t)$ is an ω-periodic continuous function on R, then

$$a(f,\lambda) = \int_0^{\omega} f(t)\mathrm{e}^{-i\lambda t}\mathrm{d}t, \tag{2.1}$$

must exist, $a(f,\lambda)$ is called the Fourier coefficient of $f(t)$, the λ such that

$a(f,\lambda)\neq 0$ is called the Fourier index of $f(t)$; there is a countable set Λ_f, when $\lambda\in\Lambda_f$, $a(f,\lambda)\neq 0$, as long as $\lambda\notin\Lambda_f$, there must be $a(f,\lambda)=0$, Λ_f is called the exponential set of $f(t)$.

Definition 2.2 (see [13]) A set of real numbers composed of linear combinations of integer coefficients of elements in Λ_f is called a module or a frequency module of $f(t)$, which is denoted as $\mathrm{mod}(f)$, that is

$$\mathrm{mod}(f)=\left\{\mu \mid \mu=\sum_{j=1}^{N} n_j\lambda_j, n_j, N\in Z^{+}, N\geq 1, \lambda_j\in\Lambda_f\right\}. \tag{2.2}$$

Lemma 2.1 (see [1]) Consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=a(t)x+b(t), \tag{2.3}$$

where $a(t),b(t)$ are ω-periodic continuous functions on R, if $\int_0^{\omega}a(t)\mathrm{d}t\neq 0$, then Equation (2.3) has a unique ω-periodic continuous solution $\eta(t)$, $\mathrm{mod}(\eta)\subseteq\mathrm{mod}(a(t),b(t))$, and $\eta(t)$ can be written as follows:

$$\eta(t)=\begin{cases}\int_{-\infty}^{t}\mathrm{e}^{\int_s^t a(\tau)\mathrm{d}\tau}b(s)\mathrm{d}s, & \int_0^{\omega}a(t)\mathrm{d}t<0\\ -\int_t^{+\infty}\mathrm{e}^{\int_s^t a(\tau)\mathrm{d}\tau}b(s)\mathrm{d}s, & \int_0^{\omega}a(t)\mathrm{d}t>0\end{cases} \tag{2.4}$$

Lemma 2.2 (see [13]) Suppose that an ω-periodic sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of R, $f(t)$ is an ω-periodic function, and $\mathrm{mod}(f_n)\subseteq\mathrm{mod}(f)(n=1,2,\cdots)$, then $\{f_n(t)\}$ is convergent uniformly on R.

Lemma 2.3 (see [14]) Suppose V is a metric space, C is a convex closed set of V, its boundary is ∂C, if $T:V\to V$ is a continuous compact mapping, such that $T(\partial C)\subseteq C$, then T has a fixed point on C.

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on R, denote

$$f_M=\sup_{t\in[0,\omega]}f(t),\ f_L=\inf_{t\in[0,\omega]}f(t). \tag{2.5}$$

3. A Unique Periodic Solution

If n is an odd number and $a(t)>0(<0), b(t)\equiv 0$, it is easy to know that Equation (1.3) has a unique periodic continuous solution $x(t)=0$; Following we discuss the case $b(t)\not\equiv 0$, and get two results about the existence of the periodic solution of Equation (1.3).

Theorem 3.1 Consider Equation (1.3), n is an odd number, $a(t),b(t)$ are ω-periodic continuous functions on R, suppose that the following conditions hold:

$$(H_1)\quad a(t)>0,$$

$$(H_2)\quad b(t)\not\equiv 0,$$

then Equation (1.3) has a unique ω-periodic continuous solution $\gamma(t)$, and

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M .$$

Proof (1) By (H_1), (H_2) and n is an odd number, according to the factorization of polynomials, we can get

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)\left(x+\sqrt[n]{\frac{b(t)}{a(t)}}\right)&\left(x^{n-1}-x^{n-2}\sqrt[n]{\frac{b(t)}{a(t)}}+x^{n-3}\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2}\right.\\ &\left.-\cdots+x^2\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}}-x\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}}+\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}}\right).\end{aligned} \tag{3.1}$$

Suppose

$$S=\{\varphi(t)\in C(R,R) \mid \varphi(t+\omega)=\varphi(t)\}. \tag{3.2}$$

Given any $\varphi(t),\psi(t)\in S$, the distance is defined as follows:

$$\rho(\varphi,\psi)=\sup_{t\in[0,\omega]}|\varphi(t)-\psi(t)|, \tag{3.3}$$

thus (S,ρ) is a complete metric space. Take a convex closed set B of S as follows:

$$B=\left\{\varphi(t)\in S \mid \left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \varphi(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \mathrm{mod}(\varphi)\subseteq \mathrm{mod}(a,b)\right\}. \tag{3.4}$$

Given any $\varphi(t)\in B$, consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)\left(x+\sqrt[n]{\frac{b(t)}{a(t)}}\right)&\left(\varphi^{n-1}(t)-\varphi^{n-2}(t)\sqrt[n]{\frac{b(t)}{a(t)}}+\varphi^{n-3}(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2}\right.\\ &\left.-\cdots+\varphi^2(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}}-\varphi(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}}+\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}}\right).\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}f(t) = a(t)&\left(\varphi^{n-1}(t)-\varphi^{n-2}(t)\sqrt[n]{\frac{b(t)}{a(t)}}+\varphi^{n-3}(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2}\right.\\ &\left.-\cdots+\varphi^2(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}}-\varphi(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}}+\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}}\right),\end{aligned} \tag{3.6}$$

then (3.5) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(t)\left(x+\sqrt[n]{\frac{b(t)}{a(t)}}\right)=f(t)x+f(t)\sqrt[n]{\frac{b(t)}{a(t)}}. \tag{3.7}$$

By (3.4) and (3.6), we have

$$\mathrm{mod}(f)\subseteq \mathrm{mod}(a,b). \tag{3.8}$$

By (H_1), (H_2), (3.4) and (3.6), we get that

$$0 < na_L \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}} \right)_L \le f(t) \le na_M \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}} \right)_M , \tag{3.9}$$

thus we have

$$\int_0^{\omega} f(t)\,\mathrm{d}t > 0. \tag{3.10}$$

Since $a(t)$, $b(t)$, $\varphi(t)$ are ω-periodic continuous functions on R, $f(t)$, $f(t)\sqrt[n]{-\frac{b(t)}{a(t)}}$ are ω-periodic continuous functions on R, by (3.10), according to Lemma 2.1, Equation (3.7) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = -\int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{3.11}$$

and

$$\mathrm{mod}(\eta) \subseteq \mathrm{mod}\left(f(t), f(t)\sqrt[n]{-\frac{b(t)}{a(t)}} \right). \tag{3.12}$$

By (3.8) and (3.12), it follows

$$\mathrm{mod}(\eta) \subseteq \mathrm{mod}(a,b). \tag{3.13}$$

By (H_1), (H_2), (3.9) and (3.11), we get

$$\begin{aligned}
\eta(t) &= -\int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{\frac{b(s)}{a(s)}}\mathrm{d}s \ge -\left(\sqrt[n]{\frac{b}{a}}\right)_M \int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\mathrm{d}s \\
&= \left(\sqrt[n]{\frac{b}{a}}\right)_M \int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t f(\tau)\mathrm{d}\tau\right) = \left(\sqrt[n]{\frac{b}{a}}\right)_M \left[\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_t^{+\infty} \\
&= \left(\sqrt[n]{\frac{b}{a}}\right)_M \left[\mathrm{e}^{\int_{+\infty}^t f(\tau)\mathrm{d}\tau} - 1\right](-\infty < t < +\infty) = -\left(\sqrt[n]{\frac{b}{a}}\right)_M = \left(\sqrt[n]{-\frac{b}{a}}\right)_L ,
\end{aligned}$$

and

$$\begin{aligned}
\eta(t) &= -\int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{\frac{b(s)}{a(s)}}\mathrm{d}s \le -\left(\sqrt[n]{\frac{b}{a}}\right)_L \int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\mathrm{d}s \\
&= \left(\sqrt[n]{\frac{b}{a}}\right)_L \int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t f(\tau)\mathrm{d}\tau\right) = \left(\sqrt[n]{\frac{b}{a}}\right)_L \left[\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_t^{+\infty} \\
&= \left(\sqrt[n]{\frac{b}{a}}\right)_L \left[\mathrm{e}^{\int_{+\infty}^t f(\tau)\mathrm{d}\tau} - 1\right](-\infty < t < +\infty) = -\left(\sqrt[n]{\frac{b}{a}}\right)_L = \left(\sqrt[n]{-\frac{b}{a}}\right)_M ,
\end{aligned}$$

hence, $\eta(t) \in B$.

Define a mapping as follows:

$$(T\varphi)(t) = -\int_t^{+\infty} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{3.14}$$

thus if given any $\varphi(t) \in B$, then $(T\varphi)(t) \in B$, hence $T : B \to B$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\varphi_k(t)\} \subseteq B (k = 1, 2, \cdots)$, then it follows

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \varphi_k(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \operatorname{mod}(\varphi_k) \subseteq \operatorname{mod}(a,b), (k = 1, 2, \cdots) \tag{3.15}$$

On the other hand, $(T\varphi_k)(t) = x_{\varphi_k}(t)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t} = a(t)\left(x_{\varphi_k}(t) + \sqrt[n]{\frac{b(t)}{a(t)}}\right)\Bigg(&\varphi_k^{n-1}(t) - \varphi_k^{n-2}(t)\sqrt[n]{\frac{b(t)}{a(t)}} + \varphi_k^{n-3}(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2} \\ &- \cdots + \varphi_k^2(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}} - \varphi_k(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}} + \sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}}\Bigg), \end{aligned} \tag{3.16}$$

thus we have

$$\left|\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right| \le 2na_M \left(\sqrt[n]{\left|\frac{b(t)}{a(t)}\right|}\right)_M \left(\sqrt[n]{\left|\frac{b(t)}{a(t)}\right|^{n-1}}\right)_M, \tag{3.17}$$

$$\operatorname{mod}\left(x_{\varphi_k}(t)\right) \subseteq \operatorname{mod}(a,b), \tag{3.18}$$

hence $\left\{\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right\}$ is uniformly bounded, therefore, $\{x_{\varphi_k}(t)\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli-arzela, for any sequence $\{x_{\varphi_k}(t)\} \subseteq B$, there exists a subsequence (also denoted by $\{x_{\varphi_k}(t)\}$) such that $\{x_{\varphi_k}(t)\}$ is convergent uniformly on any compact set of R. By (3.18), combined with Lemma 2.2, $\{x_{\varphi_k}(t)\}$ is convergent uniformly on R, that is to say, T is relatively compact on B.

Next, we prove that T is a continuous mapping.

Suppose $\{\varphi_k(t)\} \subseteq B, \varphi(t) \in B$, and

$$\varphi_k(t) \to \varphi(t), (k \to \infty) \tag{3.19}$$

Let

$$\begin{aligned} f_k(t) = a(t)\Bigg(&\varphi_k^{n-1}(t) - \varphi_k^{n-2}(t)\sqrt[n]{\frac{b(t)}{a(t)}} + \varphi_k^{n-3}(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2} \\ &- \cdots + \varphi_k^2(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}} - \varphi_k(t)\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}} + \sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}}\Bigg), \end{aligned} \tag{3.20}$$

then it follows

$$f_k(t) \to f(t), (k \to \infty) \tag{3.21}$$

and

$$0 < na_L \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L \le f_k(t) \le na_M \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_M. \tag{3.22}$$

By (3.14), we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&=\left|\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau} f_k(s)\sqrt[n]{\frac{b(s)}{a(s)}}ds-\int_t^{+\infty} e^{\int_s^t f(\tau)d\tau} f(s)\sqrt[n]{\frac{b(s)}{a(s)}}ds\right| \\
&=\left|\int_t^{+\infty}\left(e^{\int_s^t f_k(\tau)d\tau}-e^{\int_s^t f(\tau)d\tau}\right) f(s)\sqrt[n]{\frac{b(s)}{a(s)}}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau}\left(f_k(s)-f(s)\right)\sqrt[n]{\frac{b(s)}{a(s)}}ds\right| \\
&=\left|\int_t^{+\infty}\left(e^{\xi}\int_s^t\left(f_k(\tau)-f(\tau)\right)d\tau\right) f(s)\sqrt[n]{\frac{b(s)}{a(s)}}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau}\left(f_k(s)-f(s)\right)\sqrt[n]{\frac{b(s)}{a(s)}}ds\right| \\
&\le\left(\int_t^{+\infty} e^{\xi}\left|\int_s^t d\tau\right|\left|f(s)\right|\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds+\int_t^{+\infty} e^{\int_s^t f_k(\tau)d\tau}\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds\right)\rho(f_k,f),
\end{aligned}
$$

here, ξ is between $\int_s^t f_k(\tau)d\tau$ and $\int_s^t f(\tau)d\tau$, thus ξ is between $na_M\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_M(t-s)$ and $na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L(t-s)$, hence we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&\le\left(\int_t^{+\infty} e^{na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L(t-s)}\left|\int_s^t d\tau\right|\left|f(s)\right|\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L(t-s)}\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds\right)\rho(f_k,f) \\
&=\left|\int_t^{+\infty} e^{na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L(t-s)}(s-t)f(s)\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L(t-s)}\sqrt[n]{\left|\frac{b(s)}{a(s)}\right|}ds\right|\rho(f_k,f) \\
&\le\left(\frac{a_M\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_M\left(\sqrt[n]{\left|\frac{b}{a}\right|}\right)_M}{n\left(a_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L\right)^2}+\frac{\left(\sqrt[n]{\left|\frac{b}{a}\right|}\right)_M}{na_L\left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}}\right)_L}\right)\rho(f_k,f),
\end{aligned}
$$

thus we can get

$$\rho\left(T\varphi_k, T\varphi\right) \le \left(\frac{a_M \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}} \right)_M \left(\sqrt[n]{\left|\frac{b}{a}\right|} \right)_M}{n\left(a_L \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}} \right)_L \right)^2} + \frac{\left(\sqrt[n]{\left|\frac{b}{a}\right|} \right)_M}{n a_L \left(\sqrt[n]{\left(\frac{b}{a}\right)^{n-1}} \right)_L} \right) \rho\left(f_k, f\right). \quad (3.23)$$

By (3.21) and (3.23), it follows

$$\left(T\varphi_k\right)(t) \to \left(T\varphi\right)(t), \left(k \to \infty\right) \quad (3.24)$$

therefore, T is continuous. By (3.14), easy to see, $T\left(\partial B\right) \subseteq B$. According to Lemma 2.3, T has at least a fixed point on *B*, the fixed point is the ω-periodic continuous solution $\gamma(t)$ of Equation (1.3), and

$$\left(\sqrt[n]{-\frac{b}{a}} \right)_L \le \gamma(t) \le \left(\sqrt[n]{-\frac{b}{a}} \right)_M . \quad (3.25)$$

(3) We prove that Equation (1.3) has a unique periodic solution.

Let us discuss the possible range of $x(t)$ of Equation (1.3), we divide the initial value $x(t_0) = x_0$ into the following parts:

$$x_0 \in \left(-\infty, \left(\sqrt[n]{-\frac{b}{a}} \right)_L \right), \left[\left(\sqrt[n]{-\frac{b}{a}} \right)_L, \left(\sqrt[n]{-\frac{b}{a}} \right)_M \right], \left(\left(\sqrt[n]{-\frac{b}{a}} \right)_M, +\infty \right). \quad (3.26)$$

Let

$$\begin{aligned} g(t,x) &= a(t)x^n + b(t) \\ &= a(t)\left(x + \sqrt[n]{\frac{b(t)}{a(t)}} \right)\left(x^{n-1} - x^{n-2}\sqrt[n]{\frac{b(t)}{a(t)}} + x^{n-3}\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^2} \right. \\ &\quad \left. - \cdots + x^2 \sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-3}} - x\sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-2}} + \sqrt[n]{\left(\frac{b(t)}{a(t)}\right)^{n-1}} \right). \end{aligned} \quad (3.27)$$

Then we have

$$g'_x(t,x) = na(t)x^{n-1}. \quad (3.28)$$

(I) If $x_0 \in \left(-\infty, \left(\sqrt[n]{-\frac{b}{a}} \right)_L \right)$.

Consider Equation (1.3), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_0,x_0)} = g(t_0,x_0) < 0$, thus $x(t)$ may stay at $\left(-\infty, \left(\sqrt[n]{-\frac{b}{a}} \right)_L \right)$, then $\frac{\mathrm{d}x}{\mathrm{d}t} = g(t,x) < 0$, thus $x(t)$ cannot be a periodic solution of Equation (1.3).

(II) If $x_0 \in \left[\left(\sqrt[n]{-\frac{b}{a}} \right)_L, \left(\sqrt[n]{-\frac{b}{a}} \right)_M \right]$, then Equation (1.3) has an ω-periodic continuous solution $x(t)=\gamma(t)$ with initial value $x(t_0)=\gamma(t_0)$.

By (3.28), we have

$$g_x'\left(t,\sqrt[n]{-\frac{b(t)}{a(t)}}\right)>0.$$

Since

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \tag{3.29}$$

by (3.28) and (3.29), it follows

$$g_x'(t,\gamma(t))>0. \tag{3.30}$$

Now, we suppose that there is another ω-periodic continuous solution $\Psi(t)$ of Equation (1.3) which satisfies

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \Psi(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{3.31}$$

Because $g(t,x)$ is a polynomial function with continuous partial derivatives to x, Equation (1.3) satisfies the existence and uniqueness of solutions to initial value problems of differential equations, thus

$$0<\left|\gamma(t)-\Psi(t)\right|<+\infty\left(\forall t\in R\right). \tag{3.32}$$

By (3.28) and (3.31), it follows

$$g_x'(t,\Psi(t))>0. \tag{3.33}$$

Consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}\left[\gamma(t)-\Psi(t)\right]}{\mathrm{d}t}&=g(t,\gamma(t))-g(t,\Psi(t))\\&=g_x'\left[t,\Psi(t)+\theta\left(\gamma(t)-\Psi(t)\right)\right]\left(\gamma(t)-\Psi(t)\right),(0<\theta<1)\end{aligned} \tag{3.34}$$

thus we have

$$\left|\gamma(t)-\Psi(t)\right|=\left|\gamma(0)-\Psi(0)\right|\mathrm{e}^{\int_0^t g_x'\left[s,\Psi(s)+\theta(\gamma(s)-\Psi(s))\right]\mathrm{d}s}. \tag{3.35}$$

By (3.29) and (3.31), it follows

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \Psi(t)+\theta\left(\gamma(t)-\Psi(t)\right) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{3.36}$$

By (3.28) and (3.36), it follows

$$g_x'\left[t,\Psi(t)+\theta\left(\gamma(t)-\Psi(t)\right)\right]>0. \tag{3.37}$$

By (3.35) and (3.37), it follows

$$\left|\Psi(t)-\gamma(t)\right|\to+\infty,(t\to+\infty) \tag{3.38}$$

By (3.32) and (3.38), this is a contradiction, thus $\Psi(t)$ cannot be a periodic solution of Equation (1.3), that is to say, Equation (1.3) has exactly a unique

ω-periodic continuous solution $\gamma(t)$ which satisfies

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \leq \gamma(t) \leq \left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{3.39}$$

(III) If $x_0 \in \left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M, +\infty\right)$.

Consider Equation (1.3), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_0,x_0)} = g(t_0,x_0) > 0$, thus $x(t)$ may stay at $\left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M, +\infty\right)$ or $x(t) \to +\infty (t \to +\infty)$, if $x(t)$ stays at $\left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M, +\infty\right)$, we have $\frac{\mathrm{d}x}{\mathrm{d}t} = g(t,x) > 0$, then $x(t)$ cannot be a periodic solution of Equation (1.3), if $x(t) \to +\infty (t \to +\infty)$, then $x(t)$ can also not be a periodic solution of Equation (1.3).

To sum up, Equation (1.3) has a unique ω-periodic continuous solution $\gamma(t)$ which satisfies

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \leq \gamma(t) \leq \left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{3.40}$$

This is the end of the proof of Theorem 3.1.

Similarly, we can get

Theorem 3.2 Consider Equation (1.3), n is an odd number, $a(t), b(t)$ are ω-periodic continuous functions on R, suppose that the following conditions hold:

$$(H_1)\quad a(t) < 0,$$

$$(H_2)\quad b(t) \not\equiv 0,$$

then Equation (1.3) has a unique ω-periodic continuous solution $\gamma(t)$, and

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \leq \gamma(t) \leq \left(\sqrt[n]{-\frac{b}{a}}\right)_M.$$

4. Two Periodic Solutions

In this section, when $n (n \geq 2)$ is an even number, we discuss the number of periodic solutions of Equation (1.3). Since the factorization of polynomial $x^n - y^n$ varies with n when n is an even number, here we only prove the case when $n = 2^m, m \in N^+$; When n is any other even number, the results are the same as those of the following Theorem 4.1 and Theorem 4.2, the proofs are also similar as those of Theorem 4.1 and Theorem 4.2, so we omit them here, in this section, we get two results.

Theorem 4.1 Consider Equation (1.3), $n (n = 2^m, m \in N^+)$ is an even number, $a(t), b(t)$ are ω-periodic continuous functions on R, suppose that the following conditions hold:

$$(H_1)\quad a(t) > 0,$$

$$(H_2)\quad b(t) < 0,$$

then Equation (1.3) has exactly two ω-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma_1(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M,$$

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M \le \gamma_2(t) \le -\left(\sqrt[n]{-\frac{b}{a}}\right)_L.$$

Proof (1) By (H_1) and (H_2), Equation (1.3) can be written as follows:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a(t)\left(x^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(x^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots \\ &\cdot\left(x+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(x-\sqrt[n]{-\frac{b(t)}{a(t)}}\right).\end{aligned} \tag{4.1}$$

Suppose

$$S=\{\varphi(t)\in C(R,R)\,|\,\varphi(t+\omega)=\varphi(t)\}. \tag{4.2}$$

Given any $\varphi(t),\psi(t)\in S$, the distance is defined as follows:

$$\rho(\varphi,\psi)=\sup_{t\in[0,\omega]}|\varphi(t)-\psi(t)|, \tag{4.3}$$

thus (S,ρ) is a complete metric space. Take a convex closed set B_1 of S as follows:

$$B_1=\left\{\varphi(t)\in S\,|\,-\left(\sqrt[n]{-\frac{b}{a}}\right)_M \le \varphi(t) \le -\left(\sqrt[n]{-\frac{b}{a}}\right)_L, \operatorname{mod}(\varphi)\subseteq \operatorname{mod}(a,b)\right\}. \tag{4.4}$$

Given any $\varphi(t)\in B_1$, consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a(t)\left(\varphi^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots \\ &\cdot\left(x+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(\varphi(t)-\sqrt[n]{-\frac{b(t)}{a(t)}}\right).\end{aligned} \tag{4.5}$$

Let

$$f(t)=a(t)\left(\varphi^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\left(\varphi(t)-\sqrt[n]{-\frac{b(t)}{a(t)}}\right), \tag{4.6}$$

then (4.5) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(t)\left(x+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)=f(t)x+f(t)\sqrt[n]{-\frac{b(t)}{a(t)}}. \tag{4.7}$$

By (4.4) and (4.6), we have

$$\operatorname{mod}(f) \subseteq \operatorname{mod}(a,b). \tag{4.8}$$

By (H_1), (H_2), (4.4) and (4.6), we get that

$$\begin{aligned} &-2^{\log_2^n} a_M \left(\sqrt{-\frac{b}{a}}\right)_M \left(\sqrt[4]{-\frac{b}{a}}\right)_M \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_M \\ &\le f(t) \le -2^{\log_2^n} a_L \left(\sqrt{-\frac{b}{a}}\right)_L \left(\sqrt[4]{-\frac{b}{a}}\right)_L \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_L < 0, \end{aligned} \tag{4.9}$$

thus we have

$$\int_0^{\omega} f(t)\,\mathrm{d}t < 0. \tag{4.10}$$

Since $a(t)$, $b(t)$, $\varphi(t)$ are ω-periodic continuous functions on R, $f(t)$, $f(t)\sqrt[n]{-\frac{b(t)}{a(t)}}$ are ω-periodic continuous functions on R, by (4.10), according to Lemma 2.1, Equation (4.7) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s) \sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{4.11}$$

and

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}\left(f(t), f(t)\sqrt[n]{-\frac{b(t)}{a(t)}} \right). \tag{4.12}$$

By (4.4), (4.6) and (4.12), it follows

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}(a,b). \tag{4.13}$$

By (H_1), (H_2), (4.9) and (4.11), we get

$$\begin{aligned} \eta(t) &= \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s \ge \left(\sqrt[n]{-\frac{b}{a}}\right)_M \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\mathrm{d}s \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t f(\tau)\mathrm{d}\tau\right) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \left[\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_{-\infty}^{t} \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \left[1-\mathrm{e}^{\int_{-\infty}^t f(\tau)\mathrm{d}\tau}\right](-\infty < t < +\infty) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_M, \end{aligned}$$

and

$$\begin{aligned} \eta(t) &= \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s \le \left(\sqrt[n]{-\frac{b}{a}}\right)_L \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\mathrm{d}s \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \int_{-\infty}^{t} \mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t f(\tau)\mathrm{d}\tau\right) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \left[\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_{-\infty}^{t} \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \left[1-\mathrm{e}^{\int_{-\infty}^t f(\tau)\mathrm{d}\tau} - 1\right](-\infty < t < +\infty) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_L, \end{aligned}$$

hence, $\eta(t) \in B_1$.

Define a mapping as follows:

$$(T\varphi)(t)=\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{4.14}$$

thus if given any $\varphi(t)\in B_1$, then $(T\varphi)(t)\in B_1$, hence $T:B_1\to B_1$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\varphi_k(t)\}\subseteq B_1(k=1,2,\cdots)$, then it follows

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M \le \varphi_k(t)\le -\left(\sqrt[n]{-\frac{b}{a}}\right)_L, \mathrm{mod}(\varphi_k)\subseteq \mathrm{mod}(a,b),(k=1,2,\cdots) \tag{4.15}$$

On the other hand, $(T\varphi_k)(t)=x_{\varphi_k}(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}&=a(t)\left(\varphi_k^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi_k^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\\&\cdot\left(x_{\varphi_k}(t)+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(\varphi_k(t)-\sqrt[n]{-\frac{b(t)}{a(t)}}\right),\end{aligned} \tag{4.16}$$

thus we have

$$\left|\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right|\le a_M 2^{\log_2^n+1}\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M^2,$$

$$\mathrm{mod}\left(x_{\varphi_k}(t)\right)\subseteq \mathrm{mod}(a,b), \tag{4.17}$$

hence $\left\{\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right\}$ is uniformly bounded, therefore, $\{x_{\varphi_k}(t)\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli-arzela, for any sequence $\{x_{\varphi_k}(t)\}\subseteq B_1$, there exists a subsequence (also denoted by $\{x_{\varphi_k}(t)\}$) such that $\{x_{\varphi_k}(t)\}$ is convergent uniformly on any compact set of R. By (4.17), combined with Lemma 2.2, $\{x_{\varphi_k}(t)\}$ is convergent uniformly on R, that is to say, T is relatively compact on B_1.

Next, we prove that T is a continuous mapping.

Suppose $\{\varphi_k(t)\}\subseteq B_1, \varphi(t)\in B_1$, and

$$\varphi_k(t)\to\varphi(t),(k\to\infty) \tag{4.18}$$

Denote

$$f_k(t)=a(t)\left(\varphi_k^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi_k^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\left(\varphi_k(t)-\sqrt[n]{-\frac{b(t)}{a(t)}}\right), \tag{4.19}$$

then it follows

$$f_k(t)\to f(t),(k\to\infty) \tag{4.20}$$

and

$$\begin{aligned}&-2\log_2^n a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M\\&\le f_k(t)\le -2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L<0.\end{aligned}$$

By (4.14), we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&=\left|\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f_k(\tau)\mathrm{d}\tau} f_k(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s-\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right| \\
&=\left|\int_{-\infty}^{t}\left(\mathrm{e}^{\int_s^t f_k(\tau)\mathrm{d}\tau}-\mathrm{e}^{\int_s^t f(\tau)\mathrm{d}\tau}\right) f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s+\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f_k(\tau)\mathrm{d}\tau}\left(f_k(s)-f(s)\right)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right| \\
&=\left|\int_{-\infty}^{t}\left(\mathrm{e}^{\xi}\int_s^t\left(f_k(\tau)-f(\tau)\right)\mathrm{d}\tau\right) f(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right. \\
&\quad\left.+\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f_k(\tau)\mathrm{d}\tau}\left(f_k(s)-f(s)\right)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right| \\
&\le\left|\int_{-\infty}^{t}\mathrm{e}^{\xi}\int_s^t\mathrm{d}\tau\left|f(s)\right|\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s+\int_{-\infty}^{t}\mathrm{e}^{\int_s^t f_k(\tau)\mathrm{d}\tau}\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right|\rho(f_k,f),
\end{aligned}
$$

here, ξ is between $\int_s^t f_k(\tau)\mathrm{d}\tau$ and $\int_s^t f(\tau)\mathrm{d}\tau$, thus ξ is between

$$
-2\log_2^n a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M(t-s)
$$

and

$$
-2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s),
$$

hence we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&\le\left|\int_{-\infty}^{t}\mathrm{e}^{-2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\int_s^t\mathrm{d}\tau\left|f(s)\right|\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right. \\
&\quad\left.+\int_{-\infty}^{t}\mathrm{e}^{-2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right|\rho(f_k,f) \\
&=\left|\int_{-\infty}^{t}\mathrm{e}^{-2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}(t-s)\left|f(s)\right|\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right. \\
&\quad\left.+\int_{-\infty}^{t}\mathrm{e}^{-2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s\right|\rho(f_k,f) \\
&\le\left(\frac{a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{\left(2\log_2^n\left(a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)\right)^2}\right. \\
&\quad\left.+\frac{\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{2\log_2^n a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L}\right)\rho(f_k,f),
\end{aligned}
$$

thus we can get

$$\rho(T\varphi_k, T\varphi) \le \left(\frac{a_M \left(\sqrt{-\frac{b}{a}}\right)_M \left(\sqrt[4]{-\frac{b}{a}}\right)_M \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_M^2}{2\log_2^n \left(a_L \left(\sqrt{-\frac{b}{a}}\right)_L \left(\sqrt[4]{-\frac{b}{a}}\right)_L \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_L \right)^2} + \frac{\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{2\log_2^n a_L \left(\sqrt{-\frac{b}{a}}\right)_L \left(\sqrt[4]{-\frac{b}{a}}\right)_L \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_L} \right) \rho(f_k, f).$$

By (4.20) and the above inequality, it follows

$$(T\varphi_k)(t) \to (T\varphi)(t), (k \to \infty) \tag{4.21}$$

therefore, T is continuous. By (4.14), easy to see, $T(\partial B_1) \subseteq B_1$. According to Lemma 2.3, T has at least a fixed point on B_1, the fixed point is the ω-periodic continuous solution $\gamma_1(t)$ of Equation (1.3), and

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M \le \gamma_1(t) \le -\left(\sqrt[n]{-\frac{b}{a}}\right)_L. \tag{4.22}$$

(2) Suppose

$$S = \{\varphi(t) \in C(R,R) \mid \varphi(t+\omega) = \varphi(t)\}. \tag{4.23}$$

Given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t\in[0,\omega]} |\varphi(t) - \psi(t)|, \tag{4.24}$$

thus (S, ρ) is a complete metric space. Take a convex closed set B_2 of S as follows:

$$B_2 = \left\{ \varphi(t) \in S \mid \left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \varphi(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \mathrm{mod}(\varphi) \subseteq \mathrm{mod}(a,b) \right\}. \tag{4.25}$$

Given any $\varphi(t) \in B_2$, consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)\left(\varphi^{\frac{n}{2}}(t) + \sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi^{\frac{n}{4}}(t) + \sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots \cdot\left(\varphi(t) + \sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(x - \sqrt[n]{-\frac{b(t)}{a(t)}}\right). \tag{4.26}$$

Let

$$g(t) = a(t)\left(\varphi^{\frac{n}{2}}(t) + \sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi^{\frac{n}{4}}(t) + \sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\left(\varphi(t) + \sqrt[n]{-\frac{b(t)}{a(t)}}\right), \tag{4.27}$$

then (4.26) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t)\left(x - \sqrt[n]{-\frac{b(t)}{a(t)}}\right) = g(t)x - g(t)\sqrt[n]{-\frac{b(t)}{a(t)}}. \tag{4.28}$$

By (4.25) and (4.27), we have

$$\operatorname{mod}(g) \subseteq \operatorname{mod}(a,b). \tag{4.29}$$

By (H_1), (H_2), (4.25) and (4.27), we get that

$$\begin{aligned} 0 &< 2^{\log_2^n} a_L \left(\sqrt{-\frac{b}{a}}\right)_L \left(\sqrt[4]{-\frac{b}{a}}\right)_L \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_L \\ &\le g(t) \le 2^{\log_2^n} a_M \left(\sqrt{-\frac{b}{a}}\right)_M \left(\sqrt[4]{-\frac{b}{a}}\right)_M \cdots \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \end{aligned} \tag{4.30}$$

thus we have

$$\int_0^{\omega} g(t)\,\mathrm{d}t > 0. \tag{4.31}$$

Since $a(t)$, $b(t)$, $\varphi(t)$ are ω-periodic continuous functions on R, $g(t)$, $-g(t)\sqrt[n]{-\frac{b(t)}{a(t)}}$ are ω-periodic continuous functions on R, by (4.31), according to Lemma 2.1, Equation (4.28) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{4.32}$$

and

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}\left(g(t), g(t)\sqrt[n]{-\frac{b(t)}{a(t)}}\right). \tag{4.33}$$

By (4.29) and (4.33), it follows

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}(a,b). \tag{4.34}$$

By (H_1), (H_2), (4.30) and (4.32), we get

$$\begin{aligned} \eta(t) &= \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s \ge \left(\sqrt[n]{-\frac{b}{a}}\right)_L \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} g(s)\,\mathrm{d}s \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t g(\tau)\,\mathrm{d}\tau\right) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \left[\mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau}\right]_t^{+\infty} \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_L \left[\mathrm{e}^{\int_{+\infty}^t g(\tau)\mathrm{d}\tau} - 1\right](-\infty < t < +\infty) = \left(\sqrt[n]{-\frac{b}{a}}\right)_L, \end{aligned}$$

and

$$\begin{aligned} \eta(t) &= \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} g(s)\,\mathrm{d}s \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \int_t^{+\infty} \mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau} \mathrm{d}\left(\int_s^t g(\tau)\,\mathrm{d}\tau\right) = -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \left[\mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau}\right]_t^{+\infty} \\ &= -\left(\sqrt[n]{-\frac{b}{a}}\right)_M \left[\mathrm{e}^{\int_{+\infty}^t g(\tau)\mathrm{d}\tau} - 1\right](-\infty < t < +\infty) = \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \end{aligned}$$

hence, $\eta(t) \in B_2$.

Define a mapping as follows:

$$(T\varphi)(t)=\int_t^{+\infty}\mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau}g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}\mathrm{d}s, \tag{4.35}$$

thus if given any $\varphi(t)\in B_2$, then $(T\varphi)(t)\in B_2$, hence $T:B_2\to B_2$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\varphi_k(t)\}\subseteq B_2\,(k=1,2,\cdots)$, then it follows

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\le\varphi_k(t)\le\left(\sqrt[n]{-\frac{b}{a}}\right)_M,\mathrm{mod}(\varphi_k)\subseteq\mathrm{mod}(a,b),(k=1,2,\cdots) \tag{4.36}$$

On the other hand, $(T\varphi_k)(t)=x_{\varphi_k}(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}&=a(t)\left(\varphi_k^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi_k^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\\&\quad\cdot\left(\varphi_k(t)+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(x_{\varphi_k}(t)-\sqrt[n]{-\frac{b(t)}{a(t)}}\right).\end{aligned} \tag{4.37}$$

thus we have

$$\begin{gathered}\left|\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right|\le a_M\,2^{\log_2^n+1}\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M^2,\\ \mathrm{mod}\left(x_{\varphi_k}(t)\right)\subseteq\mathrm{mod}(a,b),\end{gathered} \tag{4.38}$$

hence $\left\{\frac{\mathrm{d}x_{\varphi_k}(t)}{\mathrm{d}t}\right\}$ is uniformly bounded, therefore, $\{x_{\varphi_k}(t)\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli-arzela, for any sequence $\{x_{\varphi_k}(t)\}\subseteq B_2$, there exists a subsequence (also denoted by $\{x_{\varphi_k}(t)\}$) such that $\{x_{\varphi_k}(t)\}$ is convergent uniformly on any compact set of R. By (4.38), combined with Lemma 2.2, $\{x_{\varphi_k}(t)\}$ is convergent uniformly on R, that is to say, T is relatively compact on B_2.

Next, we prove that T is a continuous mapping.

Suppose $\{\varphi_k(t)\}\subseteq B_2,\varphi(t)\in B_2$, and

$$\varphi_k(t)\to\varphi(t),(k\to\infty) \tag{4.39}$$

Denote

$$g_k(t)=a(t)\left(\varphi_k^{\frac{n}{2}}(t)+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(\varphi_k^{\frac{n}{4}}(t)+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\left(\varphi_k(t)+\sqrt[n]{-\frac{b(t)}{a(t)}}\right), \tag{4.40}$$

then it follows

$$g_k(t)\to g(t),(k\to\infty) \tag{4.41}$$

and

$$\begin{aligned}0&<2^{\log_2^n}a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L\\&\le g_k(t)\le 2^{\log_2^n}a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M.\end{aligned}$$

By (4.35), we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&=\left|\int_t^{+\infty} e^{\int_s^t g_k(\tau)d\tau} g_k(s)\sqrt[n]{-\frac{b(s)}{a(s)}}ds-\int_t^{+\infty} e^{\int_s^t g(\tau)d\tau} g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right| \\
&=\left|\int_t^{+\infty}\left(e^{\int_s^t g_k(\tau)d\tau}-e^{\int_s^t g(\tau)d\tau}\right)g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}ds+\int_t^{+\infty} e^{\int_s^t g_k(\tau)d\tau}\left(g_k(s)-g(s)\right)\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right| \\
&=\left|\int_t^{+\infty}\left(e^{\xi}\int_s^t\left(g_k(\tau)-g(\tau)\right)d\tau\right)g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{\int_s^t g_k(\tau)d\tau}\left(g_k(s)-g(s)\right)\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right| \\
&\le\left|\int_t^{+\infty} e^{\xi}\left|\int_s^t d\tau\right|\left|g(s)\right|\sqrt[n]{-\frac{b(s)}{a(s)}}ds+\int_t^{+\infty} e^{\int_s^t g_k(\tau)d\tau}\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right|\rho(g_k,g),
\end{aligned}
$$

here, ξ is between $\int_s^t g_k(\tau)d\tau$ and $\int_s^t g(\tau)d\tau$, thus ξ is between

$$2^{\log_2^n} a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M(t-s)$$

and

$$2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s),$$

hence we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \\
&\le\left|\int_t^{+\infty} e^{2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\left|\int_s^t d\tau\right|\left|g(s)\right|\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right|\rho(g_k,g) \\
&=\left|\int_t^{+\infty} e^{2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}(s-t)g(s)\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right. \\
&\quad\left.+\int_t^{+\infty} e^{2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L(t-s)}\sqrt[n]{-\frac{b(s)}{a(s)}}ds\right|\rho(g_k,g) \\
&\le\left(\frac{a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{2^{\log_2^n}\left(a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)^2}\right. \\
&\quad\left.+\frac{\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{2^{\log_2^n} a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L}\right)\rho(g_k,g),
\end{aligned}
$$

thus we can get

$$\rho\left(T\varphi_k,T\varphi\right)\le\left(\frac{a_M\left(\sqrt{-\frac{b}{a}}\right)_M\left(\sqrt[4]{-\frac{b}{a}}\right)_M\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_M^2}{2^{\log_2^n}\left(a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)^2}+\frac{\left(\sqrt[n]{-\frac{b}{a}}\right)_M}{2^{\log_2^n}a_L\left(\sqrt{-\frac{b}{a}}\right)_L\left(\sqrt[4]{-\frac{b}{a}}\right)_L\cdots\left(\sqrt[n]{-\frac{b}{a}}\right)_L}\right)\rho\left(g_k,g\right). \tag{4.42}$$

By (4.41) and (4.42), it follows

$$\left(T\varphi_k\right)(t)\to\left(T\varphi\right)(t),\left(k\to\infty\right) \tag{4.43}$$

therefore, T is continuous. By (4.35), easy to see, $T\left(\partial B_2\right)\subseteq B_2$. According to Lemma 2.3, T has at least a fixed point on B_2, the fixed point is the ω-periodic continuous solution $\gamma_2(t)$ of Equation (1.3), and

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\le\gamma_2(t)\le\left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.44}$$

(3) We prove that Equation (1.3) has exactly two periodic solutions.

Let us discuss the possible range of $x(t)$ of Equation (1.3), we divide the initial value $x(t_0)=x_0$ into the following parts:

$$x_0\in\left(-\infty,\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\right),\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L,\left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right],\left(\left(-\sqrt[n]{-\frac{b}{a}}\right)_M,\left(\sqrt[n]{-\frac{b}{a}}\right)_L\right),$$
$$\left[\left(\sqrt[n]{-\frac{b}{a}}\right)_L,\left(\sqrt[n]{-\frac{b}{a}}\right)_M\right],\left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M,+\infty\right).$$

Let

$$\begin{aligned}h(t,x)&=a(t)x^n+b(t)\\&=a(t)\left(x^{\frac{n}{2}}+\sqrt{-\frac{b(t)}{a(t)}}\right)\left(x^{\frac{n}{4}}+\sqrt[4]{-\frac{b(t)}{a(t)}}\right)\cdots\left(x+\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\left(x-\sqrt[n]{-\frac{b(t)}{a(t)}}\right)\end{aligned} \tag{4.45}$$

Then we have

$$h'_x(t,x)=na(t)x^{n-1}. \tag{4.46}$$

(I) If $x_0\in\left(-\infty,\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\right)$.

Consider Equation (1.3), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_0,x_0)}=h(t_0,x_0)>0$, thus $x(t)$ may stay at $\left(-\infty,\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\right)$ or enter into $\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L,\left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$ at some time t, if

$x(t)$ stays at $\left(-\infty,\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\right)$, then $\frac{\mathrm{d}x}{\mathrm{d}t}=h(t,x)>0$, thus $x(t)$ can not be a periodic solution of Equation (1.3), if $x(t)$ enters into $\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L,\left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$ at some time t, then there is not a $t_1\left(t_1>t_0\right)$ such that $x(t_1)=x(t_0)=x_0$, thus $x(t)$ can also not be a periodic solution of Equation (1.3).

(II) If $x_0\in\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L,\left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$, then Equation (1.3) has an ω-periodic continuous solution $x(t)=\gamma_1(t)$ with initial value $x(t_0)=\gamma_1(t_0)$.

Since $h\left(t,-\sqrt[n]{-\frac{b(t)}{a(t)}}\right)=h\left(t,\sqrt[n]{-\frac{b(t)}{a(t)}}\right)=0$, by differential mean value theorem, it follows

$$h_x'\left(t,\xi(t)\right)=0,\left(-\sqrt[n]{-\frac{b(t)}{a(t)}}<\xi(t)<\sqrt[n]{-\frac{b(t)}{a(t)}}\right).$$

By (4.46), we have

$$h_x'\left(t,-\sqrt[n]{-\frac{b(t)}{a(t)}}\right)<0, \tag{4.47}$$

$$h_x'\left(t,\sqrt[n]{-\frac{b(t)}{a(t)}}\right)>0. \tag{4.48}$$

Since

$$\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\le\gamma_1(t)\le\left(-\sqrt[n]{-\frac{b}{a}}\right)_M, \tag{4.49}$$

by (4.46) and (4.49), it follows

$$h_x'\left(t,\gamma_1(t)\right)<0. \tag{4.50}$$

Now, we suppose that there is another ω-periodic continuous solution $\Psi_1(t)$ of Equation (1.3) which satisfies

$$\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\le\Psi_1(t)\le\left(-\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.51}$$

Because $h(t,x)$ is a polynomial function with continuous partial derivatives to x, Equation (1.3) satisfies the existence and uniqueness of solutions to initial value problems of differential equations, thus

$$\left|\gamma_1(t)-\Psi_1(t)\right|>0\left(\forall t\in R\right). \tag{4.52}$$

By (4.46) and (4.51), it follows

$$h_x'\left(t,\Psi_1(t)\right)<0. \tag{4.53}$$

Consider the following equation:

$$\begin{aligned}&\frac{\mathrm{d}\left[\gamma_1(t)-\Psi_1(t)\right]}{\mathrm{d}t}=h(t,\gamma_1(t))-h(t,\Psi_1(t))\\&=h'_x\left[t,\Psi_1(t)+\theta_1(\gamma_1(t)-\Psi_1(t))\right](\gamma_1(t)-\Psi_1(t)),(0<\theta_1<1)\end{aligned} \tag{4.54}$$

thus we have

$$|\gamma_1(t)-\Psi_1(t)|=|\gamma_1(0)-\Psi_1(0)|\mathrm{e}^{\int_0^t h'_x\left[s,\Psi_1(s)+\theta_1(\gamma_1(s)-\Psi_1(s))\right]\mathrm{d}s}. \tag{4.55}$$

By (4.49) and (4.51), it follows

$$\left(-\sqrt[n]{-\frac{b}{a}}\right)_L \le \Psi_1(t)+\theta_1(\gamma_1(t)-\Psi_1(t)) \le \left(-\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.56}$$

By (4.46) and (4.56), it follows

$$h'_x\left[t,\Psi_1(t)+\theta_1(\gamma_1(t)-\Psi_1(t))\right]<0. \tag{4.57}$$

By (4.55) and (4.57), it follows

$$|\gamma_1(t)-\Psi_1(t)|\to 0,(t\to+\infty) \tag{4.58}$$

By (4.52) and (4.58), this is a contradiction, thus $\Psi_1(t)$ cannot be a periodic solution of Equation (1.3), that is to say, Equation (1.3) has exactly a unique ω-periodic continuous solution $\gamma_1(t)$ which satisfies

$$\left(-\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma_1(t) \le \left(-\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.59}$$

(III) If $x_0 \in \left(\left(-\sqrt[n]{-\frac{b}{a}}\right)_M, \left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)$.

Consider Equation (1.3), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_0,x_0)}=h(t_0,x_0)<0$, thus $x(t)$ may stay at $\left(\left(-\sqrt[n]{-\frac{b}{a}}\right)_M, \left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)$ or enter into $\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L, \left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$ at some time t, if $x(t)$ stays at $\left(\left(-\sqrt[n]{-\frac{b}{a}}\right)_M, \left(\sqrt[n]{-\frac{b}{a}}\right)_L\right)$, we have $\frac{\mathrm{d}x}{\mathrm{d}t}=h(t,x)<0$, then $x(t)$ can not be a periodic solution of Equation (1.3), if $x(t)$ enters into $\left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L, \left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$ at some time t, then there is not a $t_1(t_1>t_0)$ such that $x(t_1)=x(t_0)=x_0$, thus $x(t)$ can also not be a periodic solution of Equation (1.3).

(IV) if $x_0 \in \left[\left(-\sqrt[n]{-\frac{b}{a}}\right)_L, \left(-\sqrt[n]{-\frac{b}{a}}\right)_M\right]$, then Equation (1.3) has an ω-periodic continuous solution $x(t)=\gamma_2(t)$ with initial value $x(t_0)=\gamma_2(t_0)$.

Since

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma_2(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M, \tag{4.60}$$

by (4.46) and (4.60), it follows

$$h_x'\left(t,\gamma_2\left(t\right)\right)>0. \tag{4.61}$$

Now, we suppose that there is another ω-periodic continuous solution $\Psi_2\left(t\right)$ of Equation (1.3) which satisfies

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\leq\Psi_2\left(t\right)\leq\left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.62}$$

Because $h\left(t,x\right)$ is a polynomial function with continuous partial derivatives to *x*, Equation (1.3) satisfies the existence and uniqueness of solutions to initial value problems of differential equations, thus

$$0<\left|\gamma_2\left(t\right)-\Psi_2\left(t\right)\right|<+\infty\left(\forall t\in R\right). \tag{4.63}$$

By (4.46) and (4.62), it follows

$$h_x'\left(t,\Psi_2\left(t\right)\right)>0. \tag{4.64}$$

Consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}\left[\gamma_2\left(t\right)-\Psi_2\left(t\right)\right]}{\mathrm{d}t}&=h\left(t,\gamma_2\left(t\right)\right)-h\left(t,\Psi_2\left(t\right)\right)\\&=h_x'\left[t,\Psi_2\left(t\right)+\theta_2\left(\gamma_2\left(t\right)-\Psi_2\left(t\right)\right)\right]\left(\gamma_2\left(t\right)-\Psi_2\left(t\right)\right),\left(0<\theta_2<1\right)\end{aligned} \tag{4.65}$$

thus we have

$$\left|\gamma_2\left(t\right)-\Psi_2\left(t\right)\right|=\left|\gamma_2\left(0\right)-\Psi_2\left(0\right)\right|\mathrm{e}^{\int_0^t h_x'\left[s,\Psi_2\left(s\right)+\theta_2\left(\gamma_2\left(s\right)-\Psi_2\left(s\right)\right)\right]\mathrm{d}s}. \tag{4.66}$$

By (4.60) and (4.62), it follows

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\leq\Psi_2\left(t\right)+\theta_2\left(\gamma_2\left(t\right)-\Psi_2\left(t\right)\right)\leq\left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.67}$$

By (4.46) and (4.67), it follows

$$h_x'\left[t,\Psi_2\left(t\right)+\theta_2\left(\gamma_2\left(t\right)-\Psi_2\left(t\right)\right)\right]>0. \tag{4.68}$$

By (4.66) and (4.68), it follows

$$\left|\gamma_2\left(t\right)-\Psi_2\left(t\right)\right|\to+\infty,\left(t\to+\infty\right) \tag{4.69}$$

By (4.63) and (4.69), this is a contradiction, thus $\Psi_2\left(t\right)$ cannot be a periodic solution of Equation (1.3), that is to say, Equation (1.3) has exactly a unique ω-periodic continuous solution $\gamma_2\left(t\right)$ which satisfies

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\leq\gamma_2\left(t\right)\leq\left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.70}$$

(V) If $x_0\in\left(\left(\sqrt[n]{\frac{b}{a}}\right)_M,+\infty\right)$.

Consider Equation (1.5), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\left(t_0,x_0\right)}=h\left(t_0,x_0\right)>0$, thus $x\left(t\right)$ may stay at $\left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M,+\infty\right)$ or $x(t)\to+\infty,(t\to+\infty)$, if $x(t)$ stays at $\left(\left(\sqrt[n]{-\frac{b}{a}}\right)_M,+\infty\right)$, we have $\frac{\mathrm{d}x}{\mathrm{d}t}=h(t,x)>0$, then $x(t)$ cannot be a periodic solution of Equation (1.3), if $x(t)\to+\infty,(t\to+\infty)$, then $x(t)$ can also not be a periodic solution of Equation (1.3).

To sum up, Equation (1.3) has exactly two ω-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$ which satify

$$\left(-\sqrt[n]{-\frac{b}{a}}\right)_L\le\gamma_1(t)\le\left(-\sqrt[n]{-\frac{b}{a}}\right)_M, \tag{4.71}$$

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\le\gamma_2(t)\le\left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.72}$$

This is the end of the proof of Theorem 4.1.

Theorem 4.2 Consider Equation (1.3), $n(n\ge 2)$ is an even number, $a(t)$, $b(t)$ are ω-periodic continuous functions on R, suppose that the following conditions hold:

$$(H_1)\quad a(t)<0,$$

$$(H_2)\quad b(t)>0,$$

then Equation (1.3) has exactly two ω-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M\le\gamma_1(t)\le-\left(\sqrt[n]{-\frac{b}{a}}\right)_L,$$

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\le\gamma_2(t)\le\left(\sqrt[n]{-\frac{b}{a}}\right)_M.$$

Proof Let

$$x=-u, \tag{4.73}$$

then Equation (1.3) can be turned into the following equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}=-a(t)x^n-b(t). \tag{4.74}$$

By (H_1) and (H_2), it follows that Equation (4.74) satisfies all the conditions of Theorem 4.1, according to Theorem 4.1, Equation (4.74) has exactly two ω-periodic continuous solutions $u_1(t)$ and $u_2(t)$, and

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L\le u_1(t)\le\left(\sqrt[n]{-\frac{b}{a}}\right)_M, \tag{4.75}$$

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M\le u_2(t)\le-\left(\sqrt[n]{-\frac{b}{a}}\right)_L. \tag{4.76}$$

By (4.73), it follows that Equation (1.3) has exactly two ω-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$-\left(\sqrt[n]{-\frac{b}{a}}\right)_M \le \gamma_1(t) \le -\left(\sqrt[n]{-\frac{b}{a}}\right)_L, \tag{4.77}$$

$$\left(\sqrt[n]{-\frac{b}{a}}\right)_L \le \gamma_2(t) \le \left(\sqrt[n]{-\frac{b}{a}}\right)_M. \tag{4.78}$$

This is the end of the proof of Theorem 4.2.

5. Some Corollaries

Consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)\left(x+\gamma(t)\right)^n + b(t), \tag{5.1}$$

where $a(t), \gamma(t)$ and $b(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R.

Let

$$u = x + \gamma(t), \tag{5.2}$$

then Equation (5.1) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(t)u^n + b(t) + \frac{\mathrm{d}\gamma}{\mathrm{d}t}. \tag{5.3}$$

According to Theorem 3.1 and Theorem 3.2 of section 3, Theorem 4.1 and Theorem 4.2 of section 4, we can get:

Corollary 5.1 Consider Equation (5.1), n is an odd number, $a(t), \gamma(t), b(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R, suppose that the following condition holds:

$$(H_1) \quad a(t) \neq 0, (\forall t \in R)$$

then Equation (5.1) has a unique ω-periodic continuous solution.

Remark 5.1 If $b(t) + \frac{\mathrm{d}\gamma}{\mathrm{d}t} \equiv 0$, then Equation (5.1) has also a unique periodic solution $x(t) = 0$.

Corollary 5.2 Consider Equation (5.1), n is an even number, $a(t)$, $\gamma(t)$, $b(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R, suppose that the following condition holds:

$$(H_1) \quad a(t)\left(b(t) + \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) < 0,$$

then Equation (5.1) has exactly two ω-periodic continuous solutions.

Consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)\left(x+\gamma(t)\right)^n, \tag{5.4}$$

where $a(t)$ and $\gamma(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R.

Let

$$u = x + \gamma(t), \tag{5.5}$$

then Equation (5.5) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(t)u^n + \frac{\mathrm{d}\gamma}{\mathrm{d}t}. \tag{5.6}$$

According to Theorem 3.1 and Theorem 3.2 of section 3, we can get

Corollary 5.3 Consider Equation (5.4), n is an odd number, $a(t)$, $\gamma(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R, suppose that the following condition holds:

$$(H_1)\quad a(t) \neq 0,$$

then Equation (5.4) has a unique ω-periodic continuous solution.

Corollary 5.4 Consider Equation (5.4), n is an even number, $a(t)$, $\gamma(t)$ are ω-periodic continuous functions on R, and $\gamma(t)$ is derivable on R, suppose that the following condition holds:

$$(H_1)\quad a(t) \neq 0,$$

then Equation (5.4) has a unique ω-periodic continuous solution if and only if $\gamma(t) \equiv C$ holds.

6. Conclusions

In this paper, we get three results:

1) when n is an odd number, if $a(t) \neq 0$, then Equation (1.3) has a unique ω-periodic continuous solution, and the range of the size of the periodic continuous solution of Eq.(1.3) is also given.

2) when n is an even number, if $a(t)b(t) < 0$, then Equation (1.3) has exactly two ω-periodic continuous solutions, and the ranges of the size of the two periodic continuous solutions of Equation (1.3) are also given, one is positive, another is negative, they are symmetrical about $x = 0$.

3) We extend conclusions 1) and 2) to Equation (5.1) and Equation (5.4).

These conclusions have certain application value for judging the existence of periodic solutions of polynomial differential equations with only one higher-order term.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interest.

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