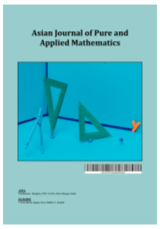

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A Theoretical Investigation on the Consistency Property of Rank-Shapley Value for Super-Additive Games

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Consistency property is an essential property of the solution concept of transferable utility games which preserves the same value for both the original game and a kind of modified games. In this work, the consistency property of Rank-Shapley value is explored through the concept of reduced game and associated game, respectively. A reduced game is a game that remains after some players have left and have been rewarded according to an established principle. Through the reduced game principle, we have demonstrated that the Rank-Shapley value admits consistency in a whole class of inessential games and null games. However, in other classes of games, it only preserves consistency for games with the number

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of players, $n \leq 2$. On the other hand, an associated game is a game generated by re-assessing (revaluing) the worth of each coalition (through a function of the worth of the coalition in the original game) such that the payoff of any player in the original game is the same as the payoff of the same player in the associated game. Here, we considered the modification of an associated game known as the Hamiache's framework. The modification generates a unique and specific associated game for any given game by adapting the proportional share of the surpluses. Hence, the Rank-Shapley value admits consistency in all classes of game through the modification of the Hamiache's framework. The consistency property of Rank-Shapley value espouses its flexibility in the theory of cooperative games involving dynamic number of players where interest is on maintaining fairness in the payoff of players. Thus, it supports the implementation of the value in a sequence (dynamic) of games.

Keywords: Associated game; reduced game: Rank-Shapley value; consistency; supper-additive games; transferable utility games.

1 Introduction

In a cooperative game theory, it is basically assumed that all the players will come together to form the grand coalition (a coalition involving all the players in the game). When the grand coalition is formed, the question remains on how to share (allocate) in a fair way, the gain of the grand coalition among the players. Shapley [1] pioneered research on the allocation rule of transferable utility (TU) games which gave birth to an allocation scheme known as the Shapley value. Kalai & Samet [2] popularized the Shapley value through a framework that allows vast interpretation of weights in sharing the dividend of a coalition. In line with the above, Eze et al. [3] proposed a solution concept known as the Rank-Shapley value for gain allocation which considers the rank of a player's stand-alone value as a sharing weight. The Rank-Shapley value presents itself as a strong alternative to the Shapley value as it is feasible in all class of transferable utility (TU) games.

In characterization of a transferable utility value function of a cooperative game, consistency is an important axiom that ensures the stability of a given solution. A solution concept (value) is consistent if it gives the same payments to players in the original game as it does to players of some kind of modified game [4]. The consistency axiom preserves the same value for both the original and the modified game. So far, there are two different ideas through which an original game can be modified to actualize consistency. These ideas are associated game idea [4], [5], [6]; and reduced game idea [7]. By associated game, the worth of each coalition in the original game is re-assessed (revalued) using an associated function (function of the worth of the coalition in the original game) to generate a new coalition worth such that the payoff of any player in the original game is the same as the payoff of the same player in the associated game. In Hamiache [5], the framework known as the Hamiache's framework assigns to every TU game a so-called associated game and then characterized Shapley value as the unique solution for TU games satisfying the inessential game property, continuity and associated consistency. This means that for every game, the Shapley value of the associated game is equal to the Shapley value of the game itself. Similarly, Xu et al. [4] applied the matrix approach for characterizing linear values of TU games in terms of associated consistency. These linear values: Equal Allocation of Non-Separable costs (EANS)-value and the Center of gravity of Imputation Set (CIS)-value were shown to be characterized by three properties: the inessential game property, continuity and associated consistency. The characterization was done through an appropriately modified notion of associated game. In the same vein, Hamiache and Navarro [8] established an axiomatic characterization of a new value for cooperative games with incomplete communication. The result was obtained by slight modifications of associated games proposed by Hamiache [5] which can be expressed as a matrix formula. A series of successive associated games were generated, and it was shown that its limit is an inessential game. It was also established that the new sharing rule coincides with the Shapley value when the communication is complete. On the other hand, the idea of reduced game tries to prove stability of a solution using a game that remains after some players have left and have been rewarded according to an established principle. A reduced game is a game that remains after some players have left and have been rewarded according to an established principle. This concept is often used in studying the characteristics of different value function in cooperative game theory. The concept of reduced games and its applications have been widely studied in literature (see for example, [9], [10], [11], etc). Specifically, Bejan et al. [12] studied the importance of reduced games in axiomatizing core extension. Their work showed the importance of the reduced game formulation and identify the corresponding converse consistency property as the differentiating characteristic between the core and its various extensions. For a class of monotonic reduced games, Llerena and Mauri [13] introduced a family of set-valued solution concepts based on consistency principles and egalitarian considerations, and studied its relation with the core. This result induces a procedure for computing the Lexmax solution for a class of games that contains games with large core. The consistency of a solution by the idea of reduced game ensures that the surviving (remaining) players in the reduced game receive the same rewards as in the original game. This approach was adopted by Hart and Mas-Colell [7] in characterizing the Shapley value. Various solutions satisfying reduced game consistency vary with respect to the definition of the reduced game [4]. In this work, we want to explore the consistency of Rank-Shapley value by considering the implementation of the value on the reduced game of Hart and Mas-Colell [7] and on the modified version of the Hamiache's framework in Hamiache [5].

2 Materials and Methods

In this section, basic definitions and notation of terms are presented. Also, the two approaches through which the consistency of Rank-Shapley is established are presented.

2.1 Basic definition and notations

A TU game on a fixed number of players N is defined by a characteristics function v. Thus, a TU game is a pair (N, v). Without loss of generality, a TU game can simply be represented as v. Let $\Omega = 2^n$ be the set of all coalitions and a subset θ of N ($\theta \in 2^n$) be a particular coalition whose size is denoted as $|\theta|$. For every coalition $\theta \in 2^n$, $v(\theta): 2^n \rightarrow \mathbb{R}$ is a function that assigns a real value (worth) to each coalition. By convention, $v(\emptyset) = 0$ where $\emptyset$ is an empty coalition. Let Ω^{N} be a collection of all games defined on a fixed set of players N. For any game $(N, v) \in \Omega^{\mathrm{N}}$, the Rank-Shapley value (RSh_i) is a solution concept that assigns a unique payoff to every individual player $i \in N$. It is given as

$$RSh_i(N, v) = \sum_{i\in\theta;\, \theta\in 2^n} \frac{r_i}{\pi_\theta} \mathrm{H}_v(\theta) \tag{1}$$

where r_i is the rank of player i stand-alone value, $H_v(\theta) = \sum_{T\subseteq\theta} (-1)^{|\theta|-t}\, v(T)$ is the dividend [14] accrued to coalition θ and π_θ is the sum of the ranks of players in θ. Equation (1) can be regarded as the Rank-Shapley value in dividend form. This is an aspect of weighted Shapley value that shares the dividend of cooperation based on the proportion of players' ranks [3].

From equation (1),

$$Rsh_i(N, v) = \sum_{i\in\theta;\, \theta\in 2^n} \left[\sum_{T\subseteq\theta} (-1)^{|\theta|-t}\, v(T) \right] \frac{r_i}{\pi_\theta}$$

Let the sub-coalition T be partitioned into two regions: $i \in T$ and $i \notin T$. Then,

$$\begin{aligned} Rsh_i(N, v) &= \sum_{i\in\theta;\, \theta\in 2^n} \left[\sum_{T\subseteq\theta, i\in T} (-1)^{|\theta|-t}\, v(T) + \sum_{T\subseteq\theta, i\notin T} (-1)^{|\theta|-t}\, v(T) \right] \frac{r_i}{\pi_\theta} \\ &= \sum_{i\in\theta;\, \theta\in 2^n} \left[\sum_{T\subseteq\theta, i\in T} (-1)^{|\theta|-t}\, v(T) - \sum_{T\subseteq\theta, i\in T} (-1)^{|\theta|-t}\, v(T-i) \right] \frac{r_i}{\pi_\theta} \\ &= \sum_{i\in\theta;\, \theta\in 2^n} \left[\sum_{T\subseteq\theta, i\in T} (-1)^{|\theta|-t}\, [v(T) - v(T-i)] \right] \frac{r_i}{\pi_\theta} \end{aligned}$$

$$Rsh_i(N,v) = r_i \sum_{i\in T} \rho_T[v(T) - v(T-i)] \tag{2}$$

Where $\rho_T = \frac{\sum_{\substack{T\subseteq\theta \\ \theta\in 2^n}} (-1)^{|\theta|-t}}{\pi_\theta}$

Equation (2) is the Rank-Shapley value in coalitional form.

3 Consistency Analysis of Rank-Shapley Value

Here, we ascertain the consistency property of the value through the idea of reduced game. First, let us consider the null game property (and its extension) of the value.

3.1 Null game (NG) property

A (perfect) null game $(N, \mathbf{0})$ is a game in which $v(\theta) = 0 \;\forall\; \theta \in 2^n$. Consequently, $Rsh_i(N, \mathbf{0}) = 0 \;\forall\; i \in N$. Now, we make a modification of the null game by restricting only the grand coalition to a value $v(N) \neq 0$ and call such game a partial null game, denoted by $\left(N, \mathbf{0}_{v(N)}\right)$. Let $\mathcal{L}_0$ denote the class of all partial null games. For every partial null game contained in $\mathcal{L}_0$, the Rank-Shapley value admits a symmetric formulation that is analogous to the egalitarian value(EV). Thus, for any $v \in \mathcal{L}_0$,

$$Rsh_i(N,v) \Rightarrow (EV)_i = \frac{v(N)}{n}$$

This formulation is obvious for every reduced game of $v \in \mathcal{L}_0$ and equals to zero for every k^{th}- order sub-game of $v \in \mathcal{L}_0$. Hence, consistency is preserved in every partial null game. This assertion is buttressed in proposition 3.

3.2 Reduced game consistency

Reduced game of cooperative games has become popularly useful in examining the characteristics (properties) of TU games and value functions. A reduced game is a game that remains after some players have left and have been rewarded according to an established principle. There are different forms of reduced game in literature. However, in this work, we build on the reduced game definition of Hart and Mas-Colell [7]. For any game (N, v), let RSh_i be a Rank-Shapley value for player i defined on a collection of games Ω^{N}. Let (S, v_S^R) be a reduced game defined on $\mathrm{S} \subset \mathrm{N}$. For any coalition $\theta \subseteq \mathrm{S}$,

$$v_S^R(\theta) = v(\theta \cup S') - \sum_{i\in S'} RSh_i(\theta \cup S', v) \tag{3}$$

where $S' = N \backslash S$.

Recall that the Rank-Shapley value is Pareto-optimal (efficient), thus,

$$\sum_{i\in S'} RSh_i(\theta \cup S', v) + \sum_{j\in\theta} RSh_j(\theta \cup S', v) = v(\theta \cup S')$$

The re-evaluation function in equation (3) reduces to

$$v_S^R(\theta) = v(\theta \cup S') - \left[v(\theta \cup S') - \sum_{j\in\theta} RSh_j(\theta \cup S', v)\right] = \sum_{j\in\theta} RSh_j(\theta \cup S', v)$$

The idea of reduced game consistency tries to ascertain whether the surviving players in a game will retain their payoff as before if the number of players in the game is reduced by an integer, S'. This central idea will guide our discussions as we consider the propositions in the following sections.

Proposition 1. Every k^{th}- order sub-game of $v \in \mathcal{L}_0$ is a null game and every reduced game of $v \in \mathcal{L}_0$ is closed in $\mathcal{L}_0$.

***Proof*:**

The proof of the first part of proposition 1 is trivial. To prove the second part of the proposition, we consider a reduced game by Hart and Mas-Colell [7] specified in equation (3).

Based on equation (3), for every partial null game $v \in \mathcal{L}_0$,

$v_S^R(\theta) = v(\theta \cup S') - \sum_{i \in S'} RSh_i(\theta \cup S', v) = 0 \ \forall \ \theta \neq N \backslash S'$ since every sub-game $(\{\theta \cup S'\}, v)$ is a null game. However, for $\theta = N \backslash S'$, $v_S^R(\theta) = v(N) - \sum_{i \in S'} RSh_i(N, v) \neq 0$. Therefore, it holds that $v_S^R(\theta) = 0 \ \forall \ \theta \neq S$ and $v_S^R(\theta) \neq 0 \ \forall \ \theta = S$. Thus, $v_S^R(\theta) \in \mathcal{L}_0 \ \forall \ S \subset N$.

Also, based on equation (3), $v_S^R(\theta) = v(\theta)$ for all $\theta \subseteq$ S if S is a set of symmetric players and S' is a set containing all the dummy players in the game, (N, v). This is explicitly, shown as follows:

Recall that if player i is a dummy player, $v(\theta \cup \{i\}) = v(\theta)$ and $RSh_i(\theta \cup \{i\}, v) = 0$ for all $\theta \in 2^n$. Using equation (3),

$$\begin{aligned} v_S^R(\theta) &= v(\theta \cup S') - \sum_{i \in S'} RSh_i(\theta \cup S', v) \\ &= v(\theta \cup S') \\ &= v(\theta) \end{aligned}$$

The implication of the above is that restricting (reducing) a game involving dummy players to only the non-dummy players will not create a change in the worth of all coalitions $\theta \subseteq S$.

***Proposition* 2**: For any two-player game (N, v), the Rank-Shapley value is a consistent value.

***Proof*:**

It is obvious that every two player game is only reducible by $S' \colon |S'| = 1$.

Let the set of the two players be $N = \{i, j\}$.

Define a reduced game $(\{i, j\} \backslash \{j\}, v_i^R)$ by v_i^R.

By consistency property, $RSh_i(\{i, j\} \backslash \{j\}, v_i^R) = RSh_i(\{i, j\}, v)$.

Specifically, $RSh_i(\{i, j\} \backslash \{j\}, v_i^R) = RSh_i(\{i\}, v_i^R) = v_i^R$.

From equation (3),

$$\begin{aligned} v_i^R &= v(i, j) - RSh_j(\{i, j\}, v) \\ &= v(i, j) - v(j) - \frac{r_j}{r_i + r_j}[v(i, j) - v(i) - v(j)] \\ &= v(i) + \frac{r_i}{r_i + r_j}[v(i, j) - v(i) - v(j)] \\ &= RSh_i(\{i, j\}, v). \end{aligned}$$

This completes the proof.

However, for $n \geq 3$, the consistency property of Rank-Shapley value using the reduced game in equation (3) fails to hold in all class of games. This can be illustrated through the following example.

Example 1.

	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(\theta)$	5	4	8	9	13	12	20
$R(\theta)$	2	1	3	3	5	4	6
$H_v(\theta)$	5	4	8	0	0	0	3

$$RSh_i(N, v) = \left(6, \quad 4 + \frac{1}{2}, \quad 8 + \frac{3}{2}\right)$$

Let $S' = \{2\}$. Consider the reduced game $v^R_{\{1,3\}}$.

$$v^R_{\{1,3\}}(1) = v(1,2) - RSh_2(\{1,2\}, v) = 9 - 4 = 5$$
$$v^R_{\{1,3\}}(3) = v(2,3) - RSh_2(\{2,3\}, v) = 12 - 4 = 8$$
$$v^R_{\{1,3\}}(1,3) = v(1,2,3) - RSh_2(\{1,2,3\}, v) = 20 - 4 - \frac{1}{2} = \frac{31}{2}$$

The Rank-Shapley value of the above reduced game is $RSh_i\left(\{1,3\}, v^R_{\{1,3\}}\right) = \left(5 + \frac{2.5}{3},\ 8 + \frac{5}{3}\right)$

Now, $RSh_i(N, v) \neq RSh_i\left(\{1,3\}, v^R_{\{1,3\}}\right)$. This justifies the claim that the Rank-Shapley value does not satisfy the consistency property in all class of game for $n \geq 3$. This is as a result of the fact that the sharing weight (rank) of the surviving players changes automatically as the game reduces. To make amend for this, we restrict consistency to a class of inessential games; hence the following definition and proposition.

Definition 1. A game is said to be inessential if $v(\theta) = \sum_{i \in \theta} v(i)$ for all θ. Let $\mathcal{B}_0$ denote a class of all inessential games. The following proposition holds:

Proposition 3. *Rank-Shapley value is a consistent value* for $\mathcal{B}_0$ *and* $\mathcal{L}_0$.

Proof:

Recall that for any $v \in \mathcal{B}_0$, $RSh_i(N, v) = v(i)$ since $H_v(\theta) = 0\ \forall\ |\theta| \geq 2$. We want to show that for any $v \in \mathcal{B}_0$, $RSh_i(\mathrm{S}, v^R_S) = RSh_i(N, v) = v(i)$ for all $i \in \mathrm{S}$. It has been shown in Eze et al. [3] that Rank-Shapley value satisfies the inessential game property. Hence, it follows that the Rank-Shapley value on any game $v \in \mathcal{B}_0$ induces a reduced game $v^R_S \in \mathcal{B}_0$ for $S \subset N$. Recall that the implication of efficiency property on equation (3) has it that $v^R_S(\theta) = \sum_{i \in \theta} RSh_i(\theta \cup S', v)$. Then, for each $i \in (\theta \cup S')$, $RSh_i(\theta \cup S', v) = v(i)$. Therefore, $v^R_S(\theta) = \sum_{i \in \theta} v(i)$ with the implication that $RSh_i(\mathrm{S}, v^R_S) = RSh_i(N, v) = v(i)$.

To prove consistency for $v \in \mathcal{L}_0$, we consider an $\{N\}$-unanimity game which is a unit basis for all the games in $\mathcal{L}_0$. Let $U_{\{N\}}$ be an $\{N\}$-unanimity game defined as follows:

$$U_{\{N\}}(\theta) = \begin{cases} 0\ for\ all\ \theta \neq N \\ 1\ for\ \theta = N \end{cases} \tag{4}$$

Considering equation (4), $RSh_i(N, U) = \frac{r_i}{\pi_N} U_{\{N\}}(N) = \frac{r_i}{\pi_N} = \frac{1}{n}$

For any reduced game (S, U_S^R) defined on S,

$$U_S^R(\theta) = \begin{cases} U(S' \cup \theta) - \sum_{i \in S'} RSh_i(\{S' \cup \theta\}, U) = 0;\ \forall\, \theta \in 2^{n-s'},\ \theta \neq S \\ U(S' \cup \theta) - \sum_{i \in S'} RSh_i(\{S' \cup \theta\}, U) = 1 - \frac{\sum_{i \in S'} r_i}{\pi_N};\ \ \theta = S \end{cases} \tag{5}$$

In the first line of equation (5), $U(S' \cup \theta) = 0$ for all $\theta \neq S$ while $RSh_i(\{S' \cup \theta\}, U) = 0$ for all $\theta \neq S$ since every sub-game $(\{S' \cup \theta\}, U)$ in the unanimity game is a perfect null game.

By consistency property,

$$RSh_j(S, U_S^R) = RSh_j(N, U) = \frac{r_j}{\pi_N} = \frac{1}{n}\ \forall\, j \in S \tag{6}$$

From the extreme LHS of equation (6),

$$\begin{aligned} RSh_j(S, U_S^R) &= \sum_{j \in \theta;\, \theta \in 2^{n-s'}} \frac{r_j}{\pi_\theta} \mathrm{H}_{U_S^R}(\theta) = \frac{r_j}{\pi_S} U_S^R(S) \\ &= \frac{r_j}{\pi_S}\left[1 - \frac{\sum_{i \in S'} r_i}{\pi_N}\right] \end{aligned}$$

In this type of game (partial null game), the rank of each player is the average rank of all the players. That is $r_i = \frac{n+1}{2}\ \forall\, i$.

Therefore, $\frac{\sum_{i \in S'} r_i}{\pi_N} = \frac{s'}{n}$ and $\pi_S = \pi_N - \sum_{i \in S'} r_i = \frac{n+1}{2}[n - s']$.

$RSh_j(S, U_S^R) = \frac{1}{n-s'}\left[1 - \frac{s'}{n}\right] = \frac{1}{n}$ for all $j \in S$

So, for any game $v \in \mathcal{B}_0$ or $v \in \mathcal{L}_0$, and any reduced game defined as in equation (3), the payoff of any player in the original game is the same as the payoff of the same player in the reduced game.

3.3 Hamiache's framework

Hamiache [5] explored the consistency of Shapley value using an associated game given as

$$v_\lambda^*(\theta) = v(\theta) + \lambda \sum_{j \in N \setminus \theta} c_j^\theta \tag{7}$$

where $c_j^\theta = v(\theta \cup j) - v(\theta) - v(j)$ is the surplus for cooperation between θ and j, and λ is a constant. According to Xu et al. [15], this associated game may be considered as an adaptation of a given game such that it reflects an optimistic self-evaluation of worth of coalitions. In Hamiache [5], $\lambda \in [0, 1]$ is invariant (constant) across coalitions, thereby allocating the same proportion of surplus to all coalitions. This restriction paves way for many associated games with respect to the original game since there are many λ satisfying $0 \leq \lambda \leq 1$ that can make $v_\lambda^*(\theta)$ feasible. The restriction also influences the point of convergence especially in a two player game. To close up the gap caused by the above shortfall, we modify Hamiache's framework by adapting λ to recognize a unique proportion for each coalition, thereby generating a unique (only one) associated game for any given game. This modification involves rank-based-proportional share of the surplus of every possible coalition with isolated players. Thus, we introduce a unique associated game through the following proposition.

Proposition 4. Given any game, v, there exists a unique associated game, v_R such that $RSh_i(N, v) = RSh_i(N, v_R)\ \forall\, i \in N$

Proof:

Let θ be a collection of active players and let $j \in N\backslash\theta$ be denoted as an isolated player [5]. By implication, θ and j are disjoint. To generate a unique associated game for θ, we connect θ to j through a union operation and observe the following rules:

i. No cooperation among the isolated players is valid.
ii. Cooperation of players in θ is valid as well as cooperation of θ and j.
iii. For every active coalition θ, θ must cooperate with each of the isolated players and share the surplus of such cooperation in proportion to θ's rank. Thus, the associated game v_R is $v(\theta)$ plus the sum of the rank-based-proportional share of the surplus of every possible coalition with isolated coalition (player).

For $\theta \subseteq N, \theta \neq \emptyset,$

$$v_R(\theta) = \begin{cases} v(\theta) + \sum_{j \in N\backslash\theta} \chi_j^\theta c_j^\theta \ if\ \theta \neq N \\ v(N)\ otherwise \end{cases}$$

Where $\chi_j^\theta = \frac{\pi_\theta}{\pi_{(\theta\cup j)}}$ is the rank-based-proportion of θ, $\pi_\theta = \sum_{i\in\theta} r_i$ is the rank (sum) of a given coalition θ and c_j^θ is the surplus of cooperation between θ and j. For an inessential game, $v_R(\theta) = v(\theta)$ since $c_j^\theta = 0$ for all θ.

To attend to proposition 4, recall that $H_v(\theta) = \sum_{T\subseteq\theta}(-1)^{|\theta|-t}\, v(T)$. It follows that

$$H_{v_R}(\theta) = \sum_{T\subseteq\theta}(-1)^{|\theta|-t}\, v_R(T) = v_R(\theta) + \sum_{T\subset\theta}(-1)^{|\theta|-t}\, v_R(T)$$

$$= v(\theta) + \sum_{j\in N\backslash\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta}(-1)^{|\theta|-t}\left(v(T) + \sum_{j\in N\backslash T}\chi_j^T c_j^T\right)$$

$$= v(\theta) + \sum_{T\subset\theta}(-1)^{|\theta|-t}\, v(T) + \sum_{j\in N\backslash\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta}(-1)^{|\theta|-t}\left(\sum_{j\in N\backslash T}\chi_j^T c_j^T\right)$$

$$= H_v(\theta) + \textstyle\sum_{j\in N\backslash\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta}(-1)^{|\theta|-t}\left(\sum_{j\in N\backslash T}\chi_j^T c_j^T\right) \quad (8)$$

By definition,

$$RSh_i(N, v_R) = \textstyle\sum_{i\in\theta;\, \theta\in 2^n} \frac{r_i}{\pi_\theta} \mathrm{H}_{v_R}(\theta) \quad (9)$$

Substituting for $\mathrm{H}_{v_R}(\theta)$ into equation (9), we have

$$\sum_{i\in\theta;\, \theta\in 2^n} \frac{r_i}{\pi_\theta}\left[H_v(\theta) + \sum_{j\in N\backslash\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta}(-1)^{|\theta|-t}\left(\sum_{j\in N\backslash T}\chi_j^T c_j^T\right)\right]$$

$$= RSh_i(N, v) + \sum_{i\in\theta;\, \theta\in 2^n} \frac{r_i}{\pi_\theta}\left[\sum_{j\in N\backslash\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta}(-1)^{|\theta|-t}\left(\sum_{j\in N\backslash T}\chi_j^T c_j^T\right)\right]$$

For any $i \in N$,

$$\sum_{i\in\theta;\,\theta\in 2^n} \frac{r_i}{\pi_\theta}\left[\sum_{j\in N\setminus\theta} \chi_j^\theta c_j^\theta + \sum_{T\subset\theta} (-1)^{|\theta|-t}\left(\sum_{j\in N\setminus T} \chi_j^T c_j^T\right)\right] = 0$$

Therefore, $RSh_i(N, v) = RSh_i(N, v_R)$.

This proof justifies the claim that the Rank-Shapley value is a consistent value with respect to the adaptation of any TU-game into the associated game, (N, v_R).

4 Discussion

The implication of the result of proposition 1 is that restricting (reducing) a game involving dummy players to only the non-dummy players will not create a change in the worth of all coalitions $\theta \subseteq S$. In addition, since S is a set of symmetric players, the ranks of the players in S will be equal as in the main game. Also, the sharing ratio for the symmetric players in the reduced game remains $\frac{1}{|\theta|}$ as it is in the main game for any symmetric player $i \in \theta$. Therefore, $RSh_i(N, v) = RSh_i(\mathrm{S}, v_S^R)$ for all $i \in \mathrm{S}$, since $v_S^R(\theta) = v(\theta)$ and S is a set of symmetric players. This translates to the weaker version of dummy player out (**DPO**) axiom which states that the exit of a dummy player does not affect the payoff of the surviving symmetric players. It is remarkable to state that a dummy player has no influence in the dividend of a coalition in which it is a member, but it has influence in the share of the dividend of the coalition. Thus, it can either help or hurt any player in a coalition. From the result of proposition 2, the Rank-Shapley value is a consistent value for (only) any two-player game (N, v). This is as a result of the fact that the sharing weight (rank) of the surviving players changes automatically as the game reduces. The existence of consistency property on a class of inessential games and partial null games as presented in proposition 3 holds, because the divided, $H_v(\theta) = 0\ \forall\ |\theta| \geq 2$ for all inessential games, and every reduced game of a partial null game is closed in itself. Similarly, the result of Proposition 4 establishes the consistency of Rank-Shapley value based on modified Hamiache's framework. The modified associated game is a unique aspect of Hamiache's framework that involves rank-based-proportional share of the surplus of every possible coalition with isolated players**.** It has the same interpretation (philosophy) as in Hamiache [5] with a slight modification in the proportion (ratio) of the surpluses that a given coalition can get from its cooperation with each of the isolated players. So, the modified associated game re-evaluates the worth of a coalition as the sum of its original worth, and of rank-based-proportion, χ_j^θ of all the possible previous surpluses. The specific nature of the modified Hamiache's framework influences convergence as it is not predicated on arbitrary choice of λ. Through the modified version of Hamiache's framework (associated game), the Rank-Shapley value is consistent in all class of super-additive game. This is similar to the Shapley value of [1] which preserves consistency on every class of games as shown in [7]. On the contrary, the proportional Shapley value of [16] only preserves a weaker version of consistency because of the non-linear nature of it

Lastly, this work is not exhaustive as it does not capture every approach of exploring consistency as a property of a value function. A further study of the concept can reveal other forms or approaches to it. Secondly, since different weighted Shapley values such as the Shapley value, Proportional Shapley value, Rank-Shapley value, etc. are not uniformly consistent, one may decide to look into the possibility of harmonizing the consistency property perhaps, from its formulation. Hence, establishing a generic framework for consistency of weighted Shapley values.

5 Conclusion

The Rank-Shapley value has operational essence as it is feasible (defined) for any kind of cooperative game and can easily be manipulated. In a cooperative game where at least one of the players has a stand-alone worth of zero or negative value, the Rank-Shapley value suffixes to be a powerful scheme for sharing the benefit of cooperation [3]. If the stand-alone values of players are equal, all the players share the rank sum on the basis of average. This case is analogous to assigning a unit weight to an individual player. Hence, the Rank-Shapley value collapses to Shapley value if and only if the individual stand-alone values are equal. Generally, this study is an addition to the literature on cooperative game theory with emphasis on consistency of Rank-Shapley value. In this work, the consistency property of Rank-Shapley value has been presented through two approaches: the reduced game and associated game approach. The reduced game consistency of the value has been established to be restricted to two-player games except for a class of inessential games and partial null games. In section (3.2), it was shown that reduced game of any inessential game is closed in its class while partial null games admit egalitarian formulation. These unique properties make it possible for inessential games and partial null games to preserve consistency axiom for any player set, $n \geq 3$. Also, the associated game consistency of the value has been established through a modified version of Hamiache's framework. Finally, this idea of consistency is useful in real life encounter involving dynamic number of players where interest is on maintaining fairness in the payoff of players in a sequence of games. Since this work concentrates only on theoretical exploration of consistency property of Rank-Shapley value through reduced game and associated game approaches, its scope is limited. Hence, interested researchers can explore the literature further to identify other instruments that can analyze the consistency property more appropriately. New formulations and or other various forms of associated and reduced game principles like in [9] and [10] can be explored and its results compared with that of the existing ones.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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