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Zeno and the Wrong Understanding of Motion—A Philosophical-Mathematical Inquiry into the Concept of Finitude as a Peculiarity of Infinity

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Abstract

In contrast to the solutions of applied mathematics to Zeno's paradoxes, I focus on the concept of motion and show that, by distinguishing two different forms of motion, Zeno's apparent paradoxes are not paradoxical at all. Zeno's paradoxes indirectly prove that distances are not composed of extensionless points and, in general, that a higher dimension cannot be completely composed of lower ones. Conversely, lower dimensions can be understood as special cases of higher dimensions. To illustrate this approach, I consider Cantor's only apparent proof that the real numbers are uncountable. However, his widely accepted indirect proof has the disadvantage that it depends on whether there is another way to make the real numbers countable. Cantor rightly assumes that there can be no smallest number between 0 and 1, and therefore no beginning of counting. For this reason he arbitrarily lists the real numbers in order to show with his diagonal method that this list can never be complete. The situation is different if we start with the largest number between 0 and 1 (0.999…) and use the method of an inverted triangle, which can be understood as a special fractal form. Here we can construct a vertical and a horizontal stratification with which it is actually possible to construct all real numbers between 0 and 1 without exception. Each column is infinite, and each number in that column is the starting point of a new triangle, while each row is finite. Even in a simple sine curve, we experience finiteness with respect to the y-axis and infinity with respect to the x-axis. The first parts of this article show that Zeno's assumptions contradict the concept of motion as such, so it is not surprising that this misconstruction leads to contradictions. In the last part, I discuss Cantor's diagonal method and explain the method of an inverted triangle that is internally structured like a fractal by repeating this inverted triangle at each column. The consequence is that we encounter two very different methods of counting. Vertically it is continuous, horizontally it is discrete. While Frege, Tarski, Cantor, Gödel and the Vienna Circle tried to derive the higher dimension from the lower, a procedure that always leads to new contradictions and antinomies (Tarski, Russell), I take the opposite approach here, in which I derive the lower dimension from the higher. This perspective seems to fail because Tarski, Russell, Wittgenstein, and especially the Vienna Circle have shown that the completeness of the absolute itself is logically contradictory. For this reason, we agree with Hegel in assuming that we can never fully comprehend the Absolute, but only its particular manifestations—otherwise we would be putting ourselves in the place of the Absolute, or even God. Nevertheless, we can understand the Absolute in its particular expressions, as I will show with the modest example of the triangle proof of the combined horizontal and vertical countability of the real numbers, which I developed in rejection of Cantor's diagonal proof.

Keywords

Zeno, False Assumptions about Motion, Finitude, Infinity, Cantor's Diagonal Method, Inverted Triangle as a Different Method, Vertical and Horizontal Dimensions, Quantum Theory, Relativity of Space and Time Depending on Velocity

1. Introduction and More Recent Interpretations

Let's start with a relatively recent interpretation of Zeno's paradoxes. Jean Francois Lyotard, who has popularized the concept of post-modernity, has included in his "La difference" [1] an excursus on Aristotle and the "before and after" of temporal movement, which is of strategic importance for his conception. Lyotard argues with Aristotle that the "human soul" (Aristotle) or the proposition (Lyotard) divides time into before and after. Problematic for both, however, is the "now", for that is precisely what time seems to be which is bounded on both sides by a point of now ([1], p. 131).

The central problem is how to determine this now, this point of the now, when it is at the same time a limit. Lyotard again refers to Aristotle when he quotes him approvingly: Considered as a limit, the now point is not time, but happening ([1], p. 132). Lyotard concludes that Aristotle distinguishes between time, which is constituted in propositions by a before-after, and the representational event, which as such is absolutely "now". In this way, Lyotard, like Heidegger, gives priority to being over positing, to thinking (Lyotard). "It happens" as lightning happens (Heidegger)—despite all differentiations, these formulations can hardly hide their mythical (and not metaphysical, as Lyotard thinks) background. Such myths "happen" and are a necessary consequence of positions that construct continuously conceived movements out of extensionless and indivisible points [2] [3]. In the case of time, this problem becomes particularly clear: the before and after is separated by the event as a boundary point, like a flash that happens, without this boundary point belonging to time itself. Hegel, on the other hand, argues that the boundary point belongs to continuous movement.

For Friedrich Kaulbach, the concept of motion becomes the guiding principle of all basic philosophical concepts—because thinking is always "on the way" between determination and liquefaction, between defining limits and going beyond them [4]. This marks the fundamental problem: the relationship in-between "defining limits" and "going beyond limits". Kaulbach argues that the principle of motion turns out to be going beyond limits. He starts from Aristotle's proposition that all movement is out of something and into something—this is already marking the fundamental problem. But does motion begins in something and ends in something? For Kaulbach, the whence and the whither (Lyotard's "before and after") are two "outermost" between which the middle, the completion of movement, passes. As surely as there are outer limits, so surely are "inner limits" possible in Kaulbach's approach [4].

The "in-between" between the ends, between the outer limits of a movement, forms a spatially or temporally extending whole. The in-between, however, must be able to be divided into sections at any time and in any place. It must be possible, so to speak, to nest partial extensions that contain each other: thus the relation of being opposite can be asserted for every pair of points that lie on the line of a process. This means nothing else than that the "external opposition" of the opposite points A and B would reproduce itself again and again in these partial extensions. Between the presence of a property and its complete absence there would be an infinite number of intermediate stages, each of which stands in relation to the other in affirmation and negation. Also seen in this way, it is essential to assume that the (total) process is divisible: this implies that the total process contains in itself the possibility of an infinite number of limitations ([4], p. 2).

However, Kaulbach overlooks the fact that we are dealing with quite different boundaries: on the one hand, the (inner) boundaries that are crossed by the movement, and on the other hand, the (outer) boundaries between which the movement takes place. To make this clear, in the first case the boundaries are crossed, but in the second case a movement is enclosed, limited, by its boundaries. Zeno's famous paradoxes of motion deal with this relationship between the crossing of boundaries and the enclosure of motion by its external boundaries. If motion is defined by crossing boundaries, it cannot be enclosed by boundaries in the same relationship. Conceptually, there is a fundamental difference between crossing and being enclosed within boundaries. This also applies to the concept of motion, because by distinguishing different forms of motion, Zeno's paradoxes disappear. Therefore we need to distinguish two different forms of motion. Zeno's paradox does not prove that there is no motion, but only that there can be no motion in extensionless points, and the consequence to be drawn is that space and time are an expression of motion (strictly speaking, velocity determines space and time).

2. The Dichotomy Paradox

These propositions can be explained on the basis of Zeno's dichotomy paradox. It is true that Zeno's paradoxes are often evaluated in such a way that they are paradoxical only for those who are not familiar with the modern methods of mathematics. Like his other antinomies, Zeno's paradoxes are indeed "technically", in this case mathematically, solvable. However, the consequential problems of the solutions seem sometimes to be more paradoxical than the original paradox. Sainsbury, for example, judges that the full answer to Zeno's paradox of the racetrack (here called the dichotomy paradox) requires a detailed reappraisal and justification of our spatial notions. Paradox for him is the notion of a boundary of spatial facts that does not itself occupy space. The only seemingly solutions are leading to more paradoxical consequences or myths. See for example the new age mystics in the footsteps of Fritjof Capra and the usage of myths in the concepts of post-structuralism [5]. The further elaboration of spatial concepts, on the other hand, is the task Zeno sets for us, concludes Sainsbury, a task that must always be tackled anew, as every generation of philosophers dealing with time and space has rightly felt ([6], p. 35). Despite all progress in mathematics, it can be said, that Zeno's paradoxes are not wholly solved [7].

Since I'm not a mathematician, I approach these questions philosophically, pointing out that Zeno's apparent paradoxes are not paradoxes if we differentiate the concept of motion. Motion can be understood as transgressing boundaries on one side and filling the space between boundaries on the other side. Zeno obviously uses the second conceptualization to argue that there is no motion at all, but this proposition is just related to the understanding of motion as crossing boundaries. In essence, Zeno just highlighted the contrast of two different kinds of motion with the consequence to argue that there is no motion at all. In my view, following the concept of substantial motion of Mulla Sadra [8], any kind of substance must be understood as filling the space between boundaries.

Aristotle addresses Zeno's paradoxes in the context of an argument about "non-motion" [9]. The first argument against the possibility of motion is that the moving object must arrive earlier at the half than at the end. This statement can be interpreted in such a way that for a spatial distance AB the first bisection divides the distance into AC and CB. It is assumed that the continuous division of this distance always divides either the first or the second half. If only the first half is divided, the infinitely continued bisection process would represent a backward motion instead of a movement from A to B, which, however, never reaches A. If, on the other hand, only the second half is always halved, then the halving process represents an infinite forward movement, which, however, never reaches B. This is because, before reaching B, we always have to measure through half of the respective distance to B first.

It is sometimes surprising how mathematicians, in their overzealous efforts to refute the meaning of Zeno's paradoxes, forget their own insights. For example, Herbert Meschkowski summarizes Zenon's paradoxes by saying that it is inconceivable that in a finite distance the path from the start to the meeting point contains infinitely many parts ([10], p. 20). Obviously, this argument is based on the consideration that an infinite set of distances, no matter how small, must always be larger than any finite distance. But this argument is true only for an infinite set of equal distances. The situation is different if we really take Zeno's argument seriously. For Zeno pointed out in his dichotomy paradox that an infinite set of small distances cannot exceed a finite limit if the length of these distances tends progressively to 0 at the same time. The increase of even such small parts is indeed infinite. But the progressive decrease of this increase is also infinite, so that the length of the distances tends to 0 and its limit is a finite number. To clarify, if the number of distances tends to infinity, but their length tends to 0, the result is a finite number.

There is a fundamental assumption in Zeno's paradoxes that logically contradicts the concept of motion. This is based on the hypothesis that there could be motion in extensionless and indivisible points of space and time, while the concept of motion states that motion is transcending such points. In the dichotomy paradox, this assumption is formulated as the contradiction that motion is at one time in such a point and at the same time ceases in it, comes to an end. The assumption that there must be a transition of motion to extensionless and indivisible points of space and time is one of the most far-reaching metaphysical assumptions related to all determinations of motion. This is to be clarified, if the motion from A to B is considered more exactly. For the assumed movement of something from A to B, in Zeno's paradox, implies that this something is actually at rest in A when it begins in A and, this is particularly plastic, is supposed to cease in B. In this respect, within this presupposition, it is absolutely necessary that there is no movement in A and B, and that these points cannot be reached by movement either. Without the assumption of the existence of extensionless and indivisible points of time and place, Zenon's paradoxes are neither paradoxical nor antinomian. This proof is the subject of the following discussion.

Let us suppose that the motion does not stop in B as an extensionless point, but goes beyond B. In this case, an arbitrary B', which goes beyond B, is to be presupposed. In this way, we reach and go beyond B at some point. Thus, if we assume a motion from A to B that does not stop and end at B, B is always reached and exceeded. However, B' itself is not reached in this case if the motion is to stop in it. If the movement goes beyond B', a new B'' can be formed, whereby B' is also reached and exceeded in this infinite process of division. Thus, an arbitrary point B without extension can only be reached if there is no more movement in B at all.

The same resolution of Zeno of Elea's paradoxes is to be performed for the case that A and B are extended. If both are extended, they can be represented as distances A1-A2 and B1-B2. In this case, the motion between the distances A1 and B2 is infinitely divisible and passes completely through the distance from A to B. Of course, with one exception, the two expansionless "boundary points" A1 and B2 of the distances A1-A2 and B1-B2. Analogously, B2 can also be represented as a distance, e.g. as the distance B3-B4, B4 again as B5-B6, etc., each boundary point Bn as the distance Bn + 1 to Bn + 2. Under the condition of the extension of the beginning and end point of a distance, Zeno's paradoxes are again not paradoxical, the distance is completely filled in this movement—with the one restriction, however, that the boundary points of the continuous distance, which are defined as extensionless and indivisible, necessarily do not belong to the distance any more.

It looks more problematic at first with the "backward movement", *i.e.* the assumption that before we get to B we have to reach 1/2 AB. Before this is reached, 1/2 of 1/2, so 1/4 AB, would have to be reached, and so on. In general, this interpretation of Zeno's paradoxes involves a backward motion from B to A. For this backward motion, however, again an A' is to be assumed, now however before A, if the motion actually does not begin in A. In this case, too, there is a distance A1-B, where A is passed at some point in the backward movement from B to A1. The same is true for the case that A can be represented as distance A1 - A2, then again A1 as distance A3 - A4 and so on.

Another possibility of solving Zenon's paradoxes seems to be to assume that space and time points are extended, but at the same time indivisible. There would be a solution to all of Zeno's paradoxes in such a way that the process of division has a limit, so that there is a minimum length and a minimum duration that cannot be further divided into smaller sizes. There would not always be a third between two moments of space and time, each moment would have a unique predecessor and an equally unique successor: "In short, time and space are discrete". Apart from the unfamiliarity of this image, there would be no conceptual difficulty with this idea as a solution to Zeno's paradoxes of motion, it is argued ([11], p. 168).

But is this really the case? It is true that Poivedevin can reject the question of what space is between two such discrete points, since there would be no space between them. He also emphasizes that a change would indeed go from one state to the next in small jumps, but without taking an intermediate state, since there would be no intermediate state ([11], p. 169). However, this solution to the problem always produces new problems that are at least as paradoxical. For how is this "leap" to be conceived with respect to the limit of discrete (extended-indivisible) points? As in the case of continuous motion, the real problem in solving Zeno's paradoxes is the boundary. If there is neither a "space" nor an intermediate state between two discrete points, do two adjacent points have the same boundary? Or do the two points have a different boundary, but are still "close" to each other? Is there not an intermediate state between the two boundaries? Again, one can argue that this intermediate state does not belong to space, but to another dimension. But wouldn't this mysterious additional dimension be something like a continuum, against which space and time appear only discrete? This consideration becomes quite vivid if we question the example given by the proponents of discrete space and time. For in this conception, motion appears as a series of successive still images in a movie. But what are continuous in this case is not the successive sequences of still images, but the film reel, which is not seen, but which is the basis of the whole, a continuous dimension in which the discreteness of the individual images is set.

The decisive objection against discrete points is, however, that it needs a boundary, a limit, so that discrete points can be considered as smallest, not further divisible points of space and time. In order to characterize these indivisible, but extended points, we again need a limit of this extension. This limit is the all decisive problem. If this limit is extended, the question is whether they are larger or smaller than the discrete points? If this boundary is larger, there must necessarily be another boundary between it and the discrete point. Is this second boundary again extended or not? If it is also extended and larger, the problem of the boundary drawing arises up to infinity. If, however, the boundary is smaller than the discrete points, it is actually the smallest measure. But these limits are now either smaller than the postulated smallest point, if they are extended. Or they are unexpanded, which would bring the unpleasant problem that the limits of discrete space and time points would be unexpanded themselves. Also in this case the smallest discrete points of space would not be the smallest points, but their boundaries. The conclusion to be drawn from this is that the notion of *smallest discrete* points of space and time is logically contradictory in itself, since to this notion belongs the specification of a boundary, which is in every case smaller than the smallest discrete points.

In a first summary, it can be concluded that Zeno's paradoxes are actually paradoxical only under conditions that are a priori contradictory to the concept of motion. If we define motion in its two and opposite forms, then it is characterized on the one hand by the crossing of (inner) boundaries, and on the other hand by the complete filling of two (outer) boundaries. From these two determinations, on the one hand, divisibility and, on the other hand, extension (by no means only in the spatial sense) are immediately to be inferred. This necessarily leads to the opposition between divisibility and extensibility as fundamental properties of motion, on the one hand, and the presuppositions of Zeno's paradoxes of indivisibility and inextensibility, on the other hand. If we connect these two pairs of opposites, as in Zeno's paradoxes, then the contrast contained in the presuppositions must necessarily present itself as a logical contradiction.

The opposition of crossing borders and being enclosed within borders emphasized here seems to be invalidated in the practical example of an arbitrary distance. On the level of appearance it is obvious that in every empirical distance AB the two end points belong to this distance—because on this physical level there can be no points without extension. If we take an arbitrary set of points on the distance from A to B, then each of these points is to be determined as a "partial function" of the distance from A to B. Since the whole of crossed immanent limits is at the same time the entire distance from A to B, they are enclosed by the limits A and B in this practical example.

Zeno, on the other hand, emphasizes in his paradoxes that even if all inner limits of a distance AB are exceeded in the forward movement, neither B nor in the backward movement A is reached. Our conclusion is that the points A and B, as outer limits of the movement of the distance from A to B, cannot necessarily belong to this movement, to this distance itself. The fundamental question is whether in the (continuous) motion from A to B the (discrete) point A is left or B is reached. Zeno argued that if there is a continuous motion from A to B, then neither the (discrete) starting point A is left nor the (discrete) final point B is reached. Contrary to all attempts to disprove his paradoxes, Zeno is correct in that the motion neither leaves the starting point A nor reaches the end point B, because neither in A nor in B is any motion at all—otherwise it would transcend both as shown above. Zeno's mistake is only to have presented his paradoxes as a logical contradiction, because this contradiction arises only and exclusively under the assumption of extensionless and indivisible now-points.

The mathematical solution of the problem in the calculus is based on the assumption that there is no quantitative difference between the distance from A to B and the points A and B, because there will always be a point h which is even closer to B than any conceivable difference can express. But it follows only that the distance from A to B is absolutely and completely filled, so that the difference between the distance of the two points A and B and the movement from A to B, by which it is bounded, is really zero. This assumption is the precondition for the fact that A and B are expansionless, their "expansion" is zero. The difference between A and B is exactly the same in both cases, regardless of whether the expansionless start and end points belong to this distance or not. Nevertheless, the question remains whether "no spatial difference" or "no quantitative difference" at the same time means that there is no difference at all, since one can think of other forms of difference, e.g. a "logical difference". For example, one could imagine that the distance AB is "close" to the points A and B, without these being part of the distance.

The basic philosophical problem is that of thinking together continuous, extended and divisible motion in a distance and discrete, indivisible as well as extensionless points as its outer limits. The questions and conclusions arising from Zeno's paradoxes are logical and philosophical, not mathematical. While philosophical positions should not and must not contradict scientific knowledge, their *interpretation* is not definitively determined by a mathematical statement as such.

3. The Arrow Paradox

The difference between a mathematical-physical solution and the logical-philosophical interpretation seems to be particularly evident in Zeno's paradox of the flying arrow that rests. In essence, however, the problem is the same as the dichotomy paradox. While the latter is about the false assumption of an expansionless point in space, the arrow paradox assumes that the same point in time can be determined. For example, Henning Genz (Professor of Theoretical Particle Physics) argues with the assumption, whatever occupies a place of exactly its own size is at rest. Now his counterargument is that everything, whether moving or at rest, occupies a place of exactly its own size *at every instant*. For him this is not taught in the first semester of physics only because it is assumed to be trivially true, he says. If one wants to know how long a *moving* body is, one determines at the same time the coordination of its starting and end point and forms their difference—the result is the length of the body. This descriptive representation, which is not taught to the students of the first physics semester only because it is so trivial, has only one small hook. This is found in the formulation: "one determines at the same time" ([12], pp. 62-63). This may be trivial, but it is wrong. Is there one and the same time independent of its "determination"? Paradoxically, in the same work, Genz implicitly argues against Zeno with the possibility of measuring a body at "the same time" (of course to prove that Zeno's paradoxes are hopelessly outdated), only to concede later in the discussion of relativity that there can be no absolute simultaneity ([12], pp. 131-133).

Of course this is possible if we are in the range of seconds. But what if this "determination" has a duration of only fractions of seconds, so that we are in the nanosecond range or even far below? Logically, the *determination* of the start and end point of anybody requires an arbitrarily small amount of time, which is negligible in practical investigation, but without it really being equal to 0. This is not only because the comparison between the determination of the start and end point requires some time span, however, small. The time span of the comparison between the beginning and the end point of the body would only be equal to 0 if the speed of the comparison itself were infinitely fast—which it cannot be logically (and which physically corresponds to the limit of the speed of light when reading the comparison results). To put it to the core: Nobody can be in the same place at the same time—this would only be possible if there were no movement of time and space. Space and time are themselves an expression of motion.

This problem applies not only to the microscopic range, but also to the macroscopic range, for example when observing distant galaxies. Generally speaking, a body is always at its place if and only if the observation time of the comparison is negligible compared to the time in which this body executes a motion. Since every comparison between two points in time necessarily requires a tiny but time span, the beginning and the end point of a body cannot be measured at the same time. Therefore, physical measurability cannot be an argument for or against the decision whether a body is at the place it occupies.

It is true that mechanics assumes the possibility of instantaneous velocities. But even this may be due to the practical negligibility of time differences and does not represent a logical-philosophical solution. Weierstrass had proposed such a solution, but this only shifts the problem presented here to other problematic areas. This conclusion follows from calculus and continuous functions (as emphasized by Weierstrass and the "at-at-theory of motion") by pointing out that although the value of a function f (t) is constant at a given t, the function f

(t) may not be constant at t ([13], p. 3). Again, although a value can be found mathematically for this point, this does not necessarily mean that the function itself is always constant.

Finally, let us consider the problem from a point of view in which there seems to be no difference between the object and the system measuring its motion. Genz's argument that a moving body is always at the place it occupies could be true if both the measured body and the measuring instrument are moving uniformly, for example under Earth conditions. According to this reasoning, the relative motion of both bodies could be neglected. But what about the Earth's own rotation, its motion around the Sun, the motion of the solar system in the Milky Way, the relative motion of all galaxies to each other, and their escape velocity with respect to the Big Bang? In order to really argue that a moving body is at the place it occupies, all the differences in these motions would have to be neglected. But they are not uniform, there are differences. From this it follows that the motion of a body would be negligible only in a certain system of observation which is absolutely uniformly moved, so that from this point of view a moved body is at the place which it occupies. This is because in the special case of uniform motion, both bodies can be represented as absolutely stationary, and their relative motion would not only be negligible, but would actually be 0. But strictly speaking there is no absolute uniform motion of two different bodies – in practice we may neglect their difference, but logically it is not possible.

To summarize these discussions, nobody is at "its" place at any time when it is moving, but only in a space or time interval which is not absolutely determinable. In conclusion, we could say that Zeno's paradoxes don't prove that there is no motion, but on the contrary, that they can only be solved if we define motion as absolute.

4. Real Numbers as Expression of Motion

The inner limits in the movement from A to B are of course transgressed because the movement goes beyond them. Nevertheless the question remains, what they are—are they themselves without extension or are they extended? Hegel agrees with the "modern" constructivists in basic mathematics that a bounded spatial, temporal, or number continuum never consists of an actual infinite set of elements such as points, lines, surfaces, "jets," real numbers, and so on. According to Hegel, the notion of the composition of a continuum from elements that can only be thought of as limits of the continuum itself does not make sense because these limits do not exist in a certain sense ([14], p. 238, note 73). What does this assessment look like with respect to real numbers? Wolff argues unequivocally that real numbers, insofar as they appear as infinite decimal fractions, but which are not repetitive, do not "exist." Rather, they would express a mere ought, namely, the absolutely unfulfillable requirement to specify so many decimal places that one and only one particular limit is reached. The whole of today's "orthodox" analysis is based on the completely "arbitrary" determination that two sequences of irrational numbers are to be ascribed the same limit value if their members finally differ by less than <e>, where <e> is any rational number.

However, this arbitrariness of fixing is challenged by the accuracy of the real numbers in physical reality. The same real numbers with which we are used to describe the things of everyday life as well as unusually large objects remain applicable also at scales far below the atomic diameter—down to less than one hundredth of the classical diameter of an elementary particle, for instance of an electron or proton. They are valid even down to the scale of quantum gravity, *i.e.* twenty orders of magnitude smaller than an elementary particle ([14], pp. 84-85).

Even this immeasurably small number, however, cannot yet exhaust the infinity of the real numbers, so that one might be inclined to agree with the position of Michael Wolff, to the effect that the real numbers actually do not exist, their reality is a mere ought. However, this interpretation of the real numbers as mere ought, put forward by Wolf, leans too closely on the practical necessity of a rational approximation for real numbers. Since real numbers are in principle infinite, one can practically calculate only with *approximate values by* stopping the infinite process at a point possible or necessary for the context. But this *approximate value* is by no means a *limit value*. If such a rational limit of an irrational infinite decimal number existed, irrational numbers could be replaced by rational ones. It is true that any (irrational) real number can be represented as an infinite sequence of rational numbers. Thus, the real number <n> (phi): = 3.1214 … can be represented as a sequence <3/1, 31/10, 312/100, 3121/1000, 31214/10 000, …> of rational numbers ([15], p. 241). Nevertheless, (irrational) real numbers are not directly traceable to rational numbers, because this *sequence* has no *limit value*, but only an *approximate value.* If this infinite sequence would break off at any point, e.g. at the smallest possible division of the objects of the universe, it would degenerate, contrary to its intention in mathematics, to a rational number ([15], pp. 243-244). In contrast to the consequence as it is represented in Zeno's paradoxes, a real number has no limit value, but only a rational approximate value. To the contrary, Brendel is understanding the infinite as a construction of the finite and highlights recursive method [16]. I argue, that the recursive method is never resulting in understanding the whole (or even the absolute or totality) and I think that the best proof for that is Gödel's theory of incompleteness

This proposition can be interpreted in such a way that a real number is basically not representable as point, but exclusively as distance, as movement. Their incompleteness contains nothing else than that their exact determination goes beyond any representation from points without extension. The addition of infinitely many lower-dimensional entities to a higher-dimensional one is paradoxical ([17], p. 51). This is the fundamental sense of real numbers as well as the reason for their almost unlimited applicability—real numbers are determinations of motion without a rational limit [18]. In contrast, rational real numbers are determinations of extensionless indivisible points.

In the latest attempt to solve Zeno's paradoxes non-standard numbers are introduced, which seem to have no physical appearance. But this introduction of a new set of numbers to solve Zeno's paradoxes points directly to our position that the numbers of a lower dimension can be derived from those of a higher dimension as its subset, but not vice versa [19] and [20].

5. The Other Countability of the Real Numbers

Since Cantor and his indirect proof using the diagonal method, it is considered unquestionable that the real numbers between 0 and 1 are uncountably or overcountability infinite [21]. However, this indirect proof has the disadvantage that it depends on whether there is a method other than the one used by Cantor, which, conversely, does not prove the uncountability of the real numbers, but only the contradictions within the diagonal method. Cantor's apparent proof has far-reaching consequences for the philosophy of mathematics, Gödel's incompleteness theorem, and philosophy as a whole in the wake of Quantum-physics. This paper develops a method to prove the countability and completeness of the real numbers by using the model of a fractual structured inversed triangle. This method is based on a beginning, a generating principle, and the arrangement of the real numbers between 0 and 1 in a fractal structure. The completeness is proved in a generating vertical triangle system, the countability is horizontal. We do not start with the smallest real number between 0 and 1, which does not exist, but with the largest, 0.9999 … or 1.

The problems associated with "inversion" in the sciences [2], are dramatically exacerbated in the indirect proof procedure. For here the truth of the "proposition" to be proved is inferred from the provable falsity of the corresponding contradiction. This method of proof is completely unproblematic in simple cases, but cannot always be applied in complex contexts. This is because in these contexts the contradiction of a proposition to be proved is no longer a simple one, but is itself a constructed model. This problem becomes particularly clear in a fundamental and far-reaching method of proof in mathematics, the diagonal method developed by Georg Cantor. The theorem of overcountability or noncountability of real numbers derived from it is only valid until a different system of countability is found than that of the diagonal method. Such a method is presented here. The simple, but far reaching difference is, that in Cantor's only apparent proof, he is arguing that there could be no beginning of counting, because there is no smallest number between 0 and 1. He is of course right. But there is a greatest number between 0 and 1, from which we could start.

The crucial point of Cantor's proof is the assumption that an infinite set is absolute complete and yet a number can be formed that belongs to this set (a real number) on the one hand and is not contained in it on the other hand. In this respect, this method only proves that, first, an infinite set cannot be closed, but always remains open, and, second, that there is a difference between the real numbers and the natural numbers. This consists in the fact that for the natural numbers (also concerning the integers and the rational numbers) there is both a beginning and a principle of generation that guarantees the completeness of these sets of numbers, whereas for the real numbers (between 0 and 1) there seems to be neither a beginning nor a principle of generation.

We want to question the proof of the uncountability of the real numbers between 0 and 1 from another side, namely by presenting a procedure that allows the countability of this set. As we have already indicated, the uncountability of this set, as postulated by Cantor [20], is, in our opinion, mainly based on the fact that he didn't find a beginning of counting for it and also no emanating principle To illustrate this, let's start with the infinite repetition of the number 0, *i.e.* 0.0000… What would be the next smallest number? 0.10000… obviously not. To distinguish it from the number 0.000… we would have to construct an infinite sequence of 0.0000… which at some point in infinity turns into the number 1. But then this number would either be a finite number, or we could form an even smaller number by inserting another 0, and only placing the digit 1 in the next but one position. The problem of the uncountability of real numbers between 0 and 1, as Cantor saw it, is objectively based on the fact that we would have to start at infinity. This is likely to be difficult. For this reason, there is no immediate beginning at 0 for the real numbers between 0 and 1, because we cannot find a logical transition from 0 to the next smallest number.

The situation is completely different if we do not look for the smallest number in this series that follows 0 (or, if we exclude 0, the smallest number between 0 and 1), but instead start with the largest number between 0 and 1, with 0.9999… (This is completely independent of whether we equate 0.9999… with 1 or not). While in the set of real numbers between 0 and 1 there can be no smallest number that can be represented in digits (because such a number that can be represented in digits would have to start at infinity), the situation is different on the other side. There is no number in this set greater than 0.99999… the one mentioned, and none that would lie between 1 and it. This means that we have at least one end of the infinite set of real numbers between 0 and 1. This allows us to reverse the counting process and use the end, the largest possible real number between 0 and 1 as the beginning of the counting process.

The problem of the relation between countable and uncountable infinite sets is thus apparently reduced to finding a beginning of counting, a method of generating these numbers, and guaranteeing the completeness of this set of numbers. Regarding the beginning, we have chosen a different beginning for the real numbers by starting with the largest number instead of the smallest – this largest possible real number between 0 and 1 as its end is a possible beginning of counting. We also believe that we have found a generation method that is also different from that of countably infinite sets. In fact, it is a method for generating all real numbers between 0 and 1. Moreover, this generation method guarantees the completeness of this number space. If we take the three criteria of the beginning, the generation, and the completeness of the generated set, then the set of real numbers between 0 and 1 is also countably infinite, though undoubtedly in a different way than the sets previously defined as countably infinite.

What does this procedure look like concretely? We do not need Cantor's diagonal method, but the modern representation of fractals. Our thesis is that we can represent all real numbers from 1 to 0 in an inversed triangle with rows and columns, which structure a particular fractal, in simplified terms, as an infinitely branched tree, and at the same time take the largest number between 0 and 1, the number 0.9999999999… as the beginning.

Let us start with the number 0.999…, followed by

The first row, which consists of the beginning of

0.9999…, followed by 0.899…, 0.799… 0699… until 0.099…

Each number would be the basis for a new column at the next row, which consists of ten ciphers, starting always with the cipher 9

First column at the second row:

0.9999…0.98999… 0.97999… 0.96999… 0.9599… 0.0099…

Followed horizontally by the second column in the second row

0.8999… 0.8899… 0.87999… 0.80999

Equally followed horizontally in this row by the third column:

0.7999…, 0.78999… 0.77999 until… 0.7.099

/

/

0.099…, 0.0899 and so forth.

If this row is completed, we go forward to row three.

Again we start with

0.999…, but exchange always the third cipher 0.998… 0997 and so forth

Again each number in a row is the basis for a new column

How do we get to the next number? Quite simply, by differentiating the second number from the first number according to the branches of a tree.

0.99999…, 0.989999…, 0.98799… etc.

We have now found the beginning of counting, the number 0.9999, and have also developed a method for generating all numbers between 0 and 1. Despite the fractal nature of this method, how can we count the set of real numbers between 0 and 1? Quite simply by creating a ruler (in the figurative sense) on each horizontal level of fractal differentiation and counting the numbers starting with the largest number on the first row using the natural numbers. At the end of each row we proceed to the beginning of the next row.

In this way we will not only capture all numbers between 0 and 1, the set generated in this way is absolutely complete. We can also count them using the natural numbers. This is possible if we represent the real numbers as part of a comprehensive fractal that starts with the number 0.9999, writes the other numbers 0.89999…, 0.79999… to the right of it, forms the next largest possible number from each of these numbers, writes the others to the right of it, and so on. On the first level of the fractal there are 10^1 numbers, on the second level 10^2 numbers, on the third level a thousand numbers and so on. This proves that the real numbers between 0 and 1 are countable. All we need is a beginning, a complete vertical generation system, and a method of horizontal counting.

Although this approach is different from it an inversed Sierpinski or an inversed Pascal triangle might visualize the principle of the countability of the real numbers between 0 and 1. The only difference is that in the bottom row we begin with the number 0.999…, in the first following row, we replace the first digit after the decimal point with 9-0. Each of these numbers is then differentiated in the following row by replacing the second digit after the decimal point with the digits 9-0, and so on. In this "inverted Herberg-Rothe triangle", which is infinite upwards, we find 10 to the power of n numbers in each row.

Each number is to be differentiated at the next level by exchanging each number in one row by the ten next ciphers from 9-0.

Finally we need to cancel all repetitions of numbers with the exception of their first appearance.

Obviously, this is a different form of countability of the real numbers between 0 and 1 than Cantor had in mind. The generation system of these numbers begins with the largest number between 0 and 1 (regardless of whether we equate the number 0.9999 with 1 or not). In this method, each digit in a tree scheme is extended by 9-0 in the next level—this method results in a vertical structure. While Cantor's apparent proof does not succeed in counting the real numbers, because there is neither a beginning, nor a generating principle, nor real completeness, these are given with the method developed here. This means that the real numbers represent a continuum vertically, but a discontinuum horizontally.

6. Summary and Perspectives

In this article, I have shown that Zeno's paradoxes are only paradoxes if we use a wrong understanding of motion. Aristotle's idea that motion begins in something and ends in something has the consequence that there is neither motion at the beginning nor at the end—in Zeno's construction, motion must come to a standstill at the end. If, on the other hand, we clearly distinguish between two forms of motion, one that completely fills space and time in between but without reaching their limits, and a kind of motion that transcends these limits, there is no paradox at all, but only different expressions of the same motion: one horizontal and one vertical. Zeno mixes the two indifferently, so he comes to the conclusion that there is no (transgressive) motion at all, because he uses only the concept of motion as filling the space between boundaries. I would like at least to point out that we can understand finitude as a special case of infinity (e.g. at very low speeds), but never the other way around. Gödel's theory of incompleteness, Cantor's and Russell's critique of infinity, as well as the work of the Vienna School, and especially Tarski's magic trick [2] of merging the endless repetition of meta-object language with meta-metalanguage, in which the difference between meta- and object language must be formulated ([2] show that the whole cannot be understood on the basis of generalizing the particular. On the contrary, it is possible to understand finitude as a particular form of infinity by being able to neglect some of its determinations—for example, Newton's mechanics is still applicable in low-speed circumstances.

Unlike Kaulbach and Aristotle, I reject the idea that motion is something that moves from one substance to another. This assumption would imply some kind of essence that moves. But we can't determine this essence of any kind without taking into account that its determination depends on its particular motion. To repeat, it could be said that particles can be understood as high, low, turning and crossing "points" of waves, but not vice versa. If we consider the subatomic space as a kind of wave-ling space, it might even be explainable that we can only work with probabilities in this field.

I must confess that I am tempted to see the finite as the particular of the infinite, the absolute. This would be a holistic understanding of the relationship between the absolute (infinite) and the particular (finite) as we find it in parts of Islam, Buddhism, and Confucianism [3]. But we also know from the followers of the Vienna Circle and Cantor that we cannot fully recognize the Absolute. However, Hegel had already argued that in determining the Absolute, we transform it into a particular of the Absolute [21]. Mathematics and logic are often based on the recursive method of complete induction—but Gödel's incompleteness theorem proves that the recursive method can never be complete. As a consequence, we may have to accept that we have to find a balance between two apparently opposed approaches—the recursive method and, at the same time, the determination of the finite as a particular of the infinite. I know that this perspective is counterintuitive because we usually experience motion as the motion of something. But space and time, perhaps the most important concepts of philosophy, mathematics, and physics, are already pure expressions of motion, based on their differentiation as substantive motion (attraction) and transgressing motion (based on repulsion in Einstein's theory).

Without treating this perspective explicitly in this short essay, I would like to point out some perspectives: Quantum theory and related mathematics have had the upper hand on Zeno's apparent paradoxes, but in doing so they still treat space and time as absolutes [22]. As long as there is no unifying theory of quantum approach and relativity, it does not make sense to base the understanding of physics on only one approach. To the contrary, in Einstein's theory of relativity, space and time are related to velocity and thus to motion [23]. It may even be possible to conceptualize particles as high, low, and turning points, as well as crossing points of waves, but waves are not composed of particles [2].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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