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Some Lacunary Sequence Spaces of Invariant Means Defined by Musielak-Orlicz Functions

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Abstract

The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of sequences of Musielak-Orlicz functions, invariant means and lacunary convergence. We establish some inclusion relations between these spaces under some conditions. This study generalized some results [1].

Keywords: Lacunary sequence; Musielak-Orlicz function; Invariant mean
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1 Introduction

Let ω be the set of all sequences of real numbers [1] and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with $(x_k) \in \mathbb{R}$ or $\mathbb{C}$ the usual norm $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N} = 1, 2, 3.....$, the positive integers.

The idea of difference sequence spaces was first introduced by Kizmaz [18] and then the concept was generalized by Et and Ç$olak$ [7]. Later on Et and Esi [8] extended the difference sequence spaces to the sequence spaces:

$$X(\Delta_v^m) = \Big\{x = (x_k) : (\Delta_v^m x) \in X\Big\},$$

for $X = \ell_\infty,\ c$ and c_0, where $v = (v_k)$ be any fixed sequence of non zero complex numbers and $(\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$.

The generalized difference operator has the following binomial representation,

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}, \text{ for all } \ k \in \mathbb{N}.$$

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The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |v_i x_i| + \|\Delta_v^m x\|_\infty.$$

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ -mean if and only if

(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has, $x_n \geq 0$ for all n

(ii) $\phi(e) = 1, e = (1, 1, 1, ...)$

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

If $x = (x_k)$, where $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \Big\{x \in \ell_\infty : \lim_k t_{kn}(x) = l, \text{ uniformly in n}\Big\}$$

$l = \sigma - \lim x$. where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + x_{\sigma^2(n)} + ... + x_{\sigma^k(n)}}{k+1} \qquad [30].$$

In the case σ is the translation mapping $n \to n+1$, σ-mean is often called a Banach limit and V_σ the set of bounded sequences of all whose invariant means are equal is the set of almost convergent sequence (see[20]),

By Lacunary sequence $\theta = (k_r), r = 0, 1, 2...$ where $k_0 = 0$ we mean an increasing sequence of non negative integers $h_r = (k_r - k_{r-1}) \to \infty \ (r \to \infty)$. The intervals determined by θ are denoted by $I_r = [k_{r-1} - k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al [9] as follow:

$$N_\theta = \Big\{x = (x_i) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=I_r} |x_k - l| = 0 \text{ for some } \ \ell\Big\}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, \ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

It is well known that if M is convex function and $M(0) = 0$ then $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 \leq \lambda \leq 1$.

Lindenstrauss and Tzafriri [21] use the idea of Orlicz function and defined the sequence space which is called an Orlicz sequence space ℓ_M such as

$$\ell_M = \Big\{x = (x_k) : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\big(\frac{|x_k|}{\rho}\big) \le 1\Big\}.$$

Which is called an Orlicz sequence space. The ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \le p < \infty$. Later the Orlicz sequence spaces were investigated by Prashar and Choudhry [25], Maddox [22],Tripathy et al.[27-29] and many others.

2 Definitions and Notations

A sequence of function $M = (M_k)$ of Orlicz function is called a $Musielak - Orliczfunction$ $[23, 24]$. Also a Musielak -Orlicz function $\Phi = (\Phi_k)$ is called $complementaryfunction$ of a Musielak-Orlicz function M if

$$\Phi_k(t) = \sup\Big\{|t|s - M_k(s)\ : s \ge 0\Big\}, \text{ for } k = 1, 2.3..$$

For a given Musielak-Orlicz function M, the Musielak-Orlicz sequence space l_M and its subspaces $\hbar_M$ are defined as follow:

$$l_M = \Big\{x = x_k \in \omega : I_M(cx) < \infty, \text{ for some } c > 0\Big\}$$

$$\hbar_M = \Big\{x = x_k \in \omega : I_M(cx) < \infty, \text{ for all } c > 0\Big\}$$

Where I_M is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), \ \ x = (x_k) \in l_M.$$

We consider l_M equipped with the Luxemburg norm

$$\|x\| = \inf\Big\{k > 0 : I_M\big(\frac{x}{k}\big) \le 1\Big\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\Big\{\frac{1}{k}\big(1 + I_M(kx)\big) : k > 0\Big\}.$$

The main purpose of this paper is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces. Let $p = (p_k)$ denote the sequences of positive real numbers, for all $k \in \mathbb{N}$. Let $M = (M_k)$ be a Musielak-Orlicz function and $u = (u_k)$ such that $u_k \ne 0 \ \ (k = 1, 2, 3, ..)$. Let s be any real number such that $s \ge 0$. Then we define the following sequence spaces:

$$[\omega^\theta, M, p, u, s]^\infty_\sigma(\Delta^m_v) = \Big\{x = (x_k) : \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \Big[M_k\big(\frac{|t_{kn}(\Delta^m_v x_k)|}{\rho}\big)\Big]^{p_k} < \infty$$

$$\rho > 0, s \geq 0\Big\}$$

$$[\omega^\theta, M, p, u, s]_\sigma(\Delta^m_v) = \Big\{x = (x_k) : \lim_{r} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \Big[M_k\big(\frac{|t_{kn}(\Delta^m_v x_k - le)|}{\rho}\big)\Big]^{p_k} = 0$$

$$\text{for some } l, \ \rho > 0, s \geq 0\Big\}$$

$$[\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v) = \Big\{x = (x_k) : \lim_{r} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \Big[M_k\big(\frac{|t_{kn}(\Delta^m_v x_k)|}{\rho}\big)\Big]^{p_k} = 0.$$

$$\rho > 0, s \geq 0\Big\}$$

Definition 2.1 A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalar (α_k) with $|\alpha_k| \leq 1$ [16]

Definition 2.2 A sequence space E is said to be monotone if it contains the canonical pre-images of all its steps spaces, [16]

Definition 2.3 If X is a Banach space normed by $\| \ . \ \|$, then $\Delta^m(X)$ is also Banach space normed by

$$\| x \|_\Delta = \sum_{k=1}^{m} |x_k| + f(\Delta^m x)$$

Remark. The following inequality will be used throughout the paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G, \ D = \max\Big(1, 2^{G-1}\Big)$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. We have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \tag{1}$$

3 Main Results

Theorem 3.1 Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then $[\omega^\theta, M, p, u, s]^\infty_\sigma(\Delta^m_v)$, $[\omega^\theta, M, p, u, s]_\sigma(\Delta^m_v)$ and $[\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$ are linear space over the field of complex numbers.

Proof. Let $x = (x_k), y = (y_k) \in [\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$ and $\alpha, \beta \in \mathbf{C}$. In order to prove the result we need to find some ρ_3 such that,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(\alpha x_k + \beta y_k))|}{\rho_3}\big)\Big]^{p_k} = 0, \text{ uniformly in n.}$$

Since $(x_k), (y_k) \in [\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$, there exist positive ρ_1, ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(x_k)|}{\rho_1}\big)\Big]^{p_k} = 0 \text{ uniformly in n}$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(y_k))|}{\rho_2}\big)\Big]^{p_k} = 0 \text{ uniformly in n.}$$

Define $\rho_3 = \max\big(2|\alpha|\rho_1, 2|\beta|\rho_2\big)$. Since (M_k) is non decreasing and convex

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(\alpha x_k + \beta y_k))|}{\rho_3}\big)\Big]^{p_k}\\
&\le \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(\alpha x_k))|}{\rho_3} + \frac{|t_{nk}(\Delta^m_v(\beta y_k))|}{\rho_3}\big)\Big]^{p_k}\\
&\le \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\Big(\frac{|t_{nk}(\Delta^m_v x_k)|}{\rho_1} + \frac{|t_{nk}(\Delta^m_v y_k)|}{\rho_2}\Big)\Big]^{p_k}\\
&\le \frac{D}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v x_k)|}{\rho}\big)\Big]^{p_k} + \frac{D}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\Big(\frac{|t_{nk}(\Delta^m_v y_k)|}{\rho}\Big)\Big]^{p_k}\\
&\to 0, \text{ as } r\to\infty, \text{ uniformly in n.}
\end{aligned}$$

So that $(\alpha x_k) + (\beta y_k) \in [\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$. This completes the proof. Similarly, we can prove that $[\omega^\theta, M, p, u, s]_\sigma(\Delta^m_v)$ and $[\omega^\theta, M, p, u, s]^\infty_\sigma(\Delta^m_v)$ are linear spaces. □

Theorem 3.2 Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then $[\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$ is a topological linear space totalparanormed by

$$\begin{aligned}
g_\Delta(x) = &\sum_{k=1}^{m} |x_k| + \inf\Big\{\rho^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\Big(\frac{|t_{nk}(\Delta^m_v x_k)|}{\rho}\Big)\Big]^{p_k}\Big)^{1/H} \le 1\\
&\text{for some } \rho,\ r = 1.2..\Big\}
\end{aligned}$$

Proof. Clearly $g_\Delta(x) = g_\Delta(-x)$. Since $M_k(0) = 0$, for all $k \in \mathbb{N}$. we get $g_\Delta(\bar\theta) = 0$, for $x = \bar\theta$. Let$x = (x_k)$, $y = (y_k) \in [\omega^\theta, M, p, u, s]^0_\sigma(\Delta^m_v)$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_r h_r^{-1} \sum_{k\in I_r} u_k k^{-s}\Big[M_k\big(\frac{|t_{nk}(\Delta^m_v(x_k))|}{\rho_1}\big)\Big]^{p_k} \le 1 \ \ r = 1, 2, 3..$$

and

$$\sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \big(\frac{|t_{nk}(\Delta_v^m(y_k))|}{\rho_2} \big) \Big]^{p_k} \leq 1 \;\; r = 1, 2, 3..$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \big(\frac{|t_{nk}(\Delta_v^m(x_k + y_k))|}{\rho} \big) \Big]^{p_k}$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \big(\frac{|t_{nk}(\Delta_v^m(x_k))|}{\rho_1} \big) \Big]^{p_k}$$
$$+ \frac{\rho_1}{\rho_1 + \rho_2} \sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \big(\frac{|t_{nk}(\Delta_v^m y_k))|}{\rho_2} \big) \Big]^{p_k}$$
$$\leq 1.$$

Since $\rho > 0$, we have

$$g_\Delta(x + y) = \sum_{k=1}^{m} \mid x_k + y_k \mid + \inf \Big\{ \rho^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{\mid t_{nk} \Delta_v^m (x_k + y_k) \mid}{\rho} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1$$

$$\text{for some } \rho > 0, \;\; r = 1.2.. \Big\}$$

$$\leq \sum_{k=1}^{m} \mid x_k \mid + \inf \Big\{ \rho_1^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{\mid t_{nk} \Delta_v^m (x_k) \mid}{\rho_1} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1$$

$$\text{for some } \rho_1 > 0, \;\; r = 1.2.. \Big\}$$

$$+ \sum_{k=1}^{m} \mid y_k \mid + \inf \Big\{ \rho_2^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{\mid t_{nk} \Delta_v^m (y_k) \mid}{\rho_2} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1$$

$$\text{for some } \rho_2 > 0, \;\; r = 1.2.. \Big\}$$

$$g_\Delta(x + y) \leq g_\Delta(x) + g_\Delta(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be a given non zero scalar in $\mathbb{C}$. Then the continuity of the product follows from the following expression.

$$g_\Delta(\lambda x) = \sum_{k=1}^{m} \mid \lambda x_k \mid + \inf \Big\{ \rho^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{\mid t_{nk} \Delta_v^m (\lambda x_k) \mid}{\rho} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1$$

$$\text{for some } \rho > 0, \;\; r = 1.2..\Big\}$$

$$= \lambda \sum_{k=1}^{m} \mid x_k \mid + \inf \Big\{ (\mid \lambda \mid \zeta)^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{\mid t_{nk} \Delta_v^m (x_k) \mid}{\zeta} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1$$

$$\text{for some } \; \zeta > 0, \;\; r = 1.2..\Big\}$$

Where $\zeta = \frac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|)^{\sup p_r}$,

$$g_\Delta(\lambda x) = \max(1, |\lambda|)^{\sup p_r} + \inf \Big\{ \rho^{p_r/H} : \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k \Big(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho} \Big) \Big]^{p_k} \Big)^{1/H} \leq 1,$$

$$\text{for some } \rho > 0, \; r = 1.2..\Big\}.$$

This completes the proof of this theorem. □

Theorem 3.3 Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then $[\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^m) \subset [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m) \subset [\omega^\theta, M, p, u, s]_\sigma^0(\Delta_v^m)$.

Proof. The inclusion $[\omega^\theta, M, p, u, s]_\sigma^0(\Delta_v^m) \subset [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$ is obvious. Let $x_k \in [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$. Then there exists some positive number ρ_1 such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k (\frac{|t_{nk}(\Delta_v^m x_k - le)|}{\rho_1}) \Big]^{p_k} \to 0$$

as $r \to \infty$, uniformly in n. Define $\rho = 2\rho_1$. Since M_k is non decreasing and convex for all $k \in \mathbb{N}$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k (\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho}) \Big]^{p_k}$$

$$\leq \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \Big[M_k (\frac{|t_{nk}(\Delta_v^m x_k - le)|}{\rho_1}) \Big]^{p_k} + \frac{D}{h_r} \sum_{k \in I_r} \Big[M_k (\frac{|le|}{\rho_1} \Big]^{p_k}$$

$$\leq \frac{D}{h_r} \sum_{k \in I_r} [M_k (\frac{|t_{nk}(\Delta_v^m x_k - le)|}{\rho_1})]^{p_k} + D \max \Big\{ 1, \big[M (\frac{|le|}{\rho_1}) \big]^G \Big\}$$

Where $G = \sup_k(p_k)$, $D = \max\big(1, 2^G - 1\big)$ by(1).
Thus $x_k \in [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$ □

Theorem 3.4 Let $M = (M_k)$ be a Musielak-Orlicz functions. If $\sup_k \left[M_k(z)\right]^{p_k} < \infty$ for all $z > 0$, then

$$[\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m) \subset [\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^m).$$

Proof. Let $x_k \in [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$ by using(1), we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k}$$
$$\le \frac{D}{h_r}\sum_{k\in I_r}\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k - le)|}{\rho})\Big]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r}\Big[M_k(\frac{|le|}{\rho})\Big]^{p_k}$$

Since $\sup_k[M(z)]^{p_k} < \infty$, we can take the $\sup_k[M(z)]^{p_k} = K$. Hence we can get $x_k \in [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$. This complete the proof. □

Theorem 3.5 Let $m \ge 1$ be fixed integer. Then the following statements are equivalent:

(i) $[\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^m)$

(ii) $[\omega^\theta, M, p, u, s]_\sigma(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m)$

(iii) $[\omega^\theta, M, p, u, s]_\sigma^o(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s]_\sigma^0(\Delta_v^m)$.

Proof. Let $x_k \in [\omega^\theta, M, p, u, s]_\sigma^o(\Delta_v^{m-1})$. Then there exist $\rho > 0$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} \to 0.$$

Since M_k is non decreasing and convex, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{2\rho})\Big]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^{m-1}x_k - \Delta^{m-1}x_{k+1})|}{2\rho})\Big]^{p_k}$$
$$\le \frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^{m-1}x_k)|}{2\rho})\Big]^{p_k} + \frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^{m-1}x_{k+1})|}{2\rho})\Big]^{p_k}$$
$$\le \frac{D}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^{m-1}x_k)|}{\rho})\Big]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^{m-1}x_{k+1})|}{\rho})\Big]^{p_k}.$$

Taking $\lim_{r\to\infty}$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k k^{-s}\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} = 0,$$

i.e $x_k \in [\omega^\theta, M, p, u, s]_\sigma^o(\Delta_v^{m-1})$. The rest of these cases can be proved in similar way.

□

Theorem 3.6 Let $M = (M_k)$ and $T = (T_k)$ be two Musielak-Orlicz functions. Then we have

(i) $[\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^m) \cap [\omega^\theta, T, p, u, s]_\sigma^\infty(\Delta_v^m) \subset [\omega^\theta, M+T, p, u, s]_\sigma^\infty(\Delta_v^m)$

(ii) $[\omega^\theta, M, p, u, s]_\sigma(\Delta_v^m) \cap [\omega^\theta, T, p, u, s]_\sigma(\Delta_v^m) \subset [\omega^\theta, M+T, p, u, s]_\sigma(\Delta_v^m)$

(iii) $[\omega^\theta, M, p, u, s]_\sigma^0(\Delta_v^m) \cap [\omega^\theta, T, p, u, s]_\sigma^0(\Delta_v^m) \subset [\omega^\theta, M+T, p, u, s]_\sigma^0(\Delta_v^m)$.

Proof. Let $x_k \in [\omega^\theta, M, p, u, s]_\sigma^\infty(\Delta_v^m) \cap [\omega^\theta, T, p, u, s]_\sigma^\infty(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} < \infty$$

and

$$\sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[T_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} < \infty$$

uniformly in n. We have

$$\Big[(M_k+T_k)(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k}$$

$$\leq D\Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} + D\Big[T_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k}$$

by(1). Applying $\sum_{k\in I_r}$ and multiplying by u_k, $\frac{1}{h_r}$ and k^{-s} both side of this inequality, we get.

$$\frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[(M_k+T_k)(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k},$$

$$\leq \frac{D}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[M_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k} + \frac{D}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[T_k(\frac{|t_{nk}(\Delta_v^m x_k)|}{\rho})\Big]^{p_k}$$

uniformly in n. This completes the proof.(ii) and (iii) can be proved similar to (i) □

Theorem 3.7 (i) The sequence spaces $[\omega^\theta, M, p, u, s]_\sigma^\infty$ and $[\omega^\theta, M, p, u, s]_\sigma^0$ are solid and hence they are monotone.
(ii) The space $[\omega^\theta, M, p, u, s]_\sigma$ is not monotone and neither solid nor perfect.

Proof. We give the proof for $[\omega^\theta, M, p, u, s]_\sigma^0$. Let $x_k \in [\omega^\theta, M, p, u, s]_\sigma^0$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[M_k(\frac{|t_{nk}(\alpha_k x_k)|}{\rho})\Big]^{p_k} \leq \frac{1}{h_r} \sum_{k\in I_r} u_k k^{-s} \Big[M_k(\frac{|t_{nk}(x_k)|}{\rho})\Big]^{p_k} \to 0$$

$(r \to \infty)$, uniformly in n. Hence $(\alpha_k x_k) \in [\omega^\theta, M, p, u, s]_\sigma^0$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $x_k \in [\omega^\theta, M, p, u, s]_\sigma^0$. The spaces are monotone follows from the remark(1) □

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