British Journal of Mathematics & Computer Science
4(12): 1678-1681, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Singular Values of a Family of Singular Perturbed Exponential Map

Mohammad Sajid[1*]

[1] *College of Engineering, Qassim University, Buraidah, Al-Qassim, Saudi Arabia.*

Short Research Article

Received: 18 February 2014
Accepted: 15 March 2014
Published: 05 April 2014

Abstract

The singular values of the one parameter family of singular perturbed exponential map $f_\mu(z) = e^z + \frac{\mu}{z}$, μ is non-zero real, are investigated. It is found that the function $f_\mu(z)$ has infinitely many singular values and all these singular values are bounded.

Keywords: Critical values; singular values
2010 Mathematics Subject Classification: 30D05; 58K05

1 Introduction

The singular perturbation occurs when a pole is introduced into a function. The singular perturbation is introduced by Devaney et al. [1] in complex dynamics by studying the dynamics of the mapping $f_\epsilon(z) = z^2 + \frac{\epsilon}{z}$ for $\epsilon > 0$. The dynamics of one parameter exponential family $E_\lambda(z) = \lambda e^z$ is studied by Devaney [2]. Rempe [3] investigated the dynamics of the family $E_\kappa(z) = e^z + \kappa$. The Julia sets of a noise-perturbed Mandelbrot map is found in [4]. The dynamical properties of functions are crucial to determine if singular values exist. All above families of functions have finite singular values. It has become more difficult but interesting if functions have infinitely many singular values.

The complex plane and the extended complex plane are denoted by $\mathbb{C}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ respectively. A point $z^* \in \mathbb{C}$ is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. The singular values of a transcendental meromorphic function are

*Corresponding author: E-mail: msajid@qec.edu.sa

precisely the points where some branch of f^{-1} fails to be defined. A function f is called critically bounded or it is said to be a function of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type.

The Nevanlinna characteristic of a meromorphic function $f(z)$ is defined by $T(r,f) = m(r,f) + N(r,f)$, where $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi$, $N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t} dt + n(0,f)\log r$ and $n(r,f) = n(r,\infty,f)$ are the number of poles of f in the disk $|z| \leq r$, counted according to its multiplicity. The Nevanlinna order [5] ρ of the function f is defined as $\rho = \overline{\lim}_{r\to\infty} \frac{\log T(r,f)}{\log r}$. Avetision, Arakelion and Gonchar [6] proved the following result which is surety for finite number of asymptotic values of a meromorphic function:

Theorem 1. *Let f be a meromorphic function of a finite Nevanlinna order ρ and $\underline{\lim}_{r\to\infty} \frac{n(r,\infty,f)}{\log r} < +\infty$. Then, the number of finite asymptotic values of the function f counted according to their multiplicity is not greater than 2ρ.*

We know that singular values play very significant role in the dynamics of functions. For instance, the following result [7] exhibits the importance of singular values in the dynamics of a transcendental meromorphic function: Suppose z_0 lies on an attracting cycle or a parabolic cycle $f(z)$. Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_0.

For infinitely many bounded singular values, the dynamical properties of transcendental meromorphic families $\lambda\frac{\sinh^2(z)}{z^4}$ and $\lambda\frac{\cos(z)}{z}$ are studied in [8] and [9] respectively.

Our goal in the present paper is to describe the singular values of perturbed exponential map. For this purpose, the function of the form $e^z + \frac{\mu}{z}$ is considered when a pole at origin is introduced into the map e^z. Let

$$\mathcal{H} = \left\{ f_\mu(z) = e^z + \frac{\mu}{z} : \mu \in \mathbb{R} \setminus \{0\}, z \in \hat{\mathbb{C}} \right\}$$

be one parameter family of perturbed exponential map. It is clear that a small perturbation in exponential families λe^z and $e^z + \kappa$ give the function in family $\mathcal{H}$. Due to this reason, our family $\mathcal{H}$ is more interesting and important for studying the dynamical properties. The function $f_\lambda \in \mathcal{H}$ is a transcendental meromorphic function with a single pole at zero and it is neither even nor odd and not periodic.

2 Main Result

The following theorem shows that the functions in the family $\mathcal{H}$ have infinitely many bounded singular values:

Theorem 2. *Let $f_\mu \in \mathcal{H}$. Then, the function $f_\mu(z)$ possesses infinitely many singular values and all these singular values of the function $f_\mu(z)$ are bounded.*

Proof. The critical points of the function $f_\mu(z)$ are solutions of the equation $f'_\mu(z) = 0$. This implies that solutions of the equation $z^2e^z - \mu = 0$ are critical points of $f_\mu(z)$. This equation has a solution z_0 if only if the equation $w^2 - \mu e^w = 0$ has a solution $-z_0$.

Let $\mu = \frac{1}{a}$. Then, we get $aw^2 - e^w = 0$. For all w such that $|w| = R$, we have

$$|-e^w| = e^{Re(w)} < e^R < |a|R^2 = |aw^2| \quad \text{since} \quad |a| > \frac{e^R}{R^2}.$$

For $R \neq 1$, using Rouche's Theorem, the equation $aw^2 - e^w = 0$ has 2 solutions (counting multiplicity) w satisfying $|w| < R$, where $0 \neq |a| > \frac{e^R}{R^2}$.

Now, for $R = 1$, the equation $aw^2 - e^w = 0$ $(|a| > e)$ has simple two roots in $|w| < 1$. To prove this, suppose w_0 is a root of order k, $k \geq 2$, then

$$aw_0^2 - e^{w_0} = 0 \quad \text{and} \quad 2aw_0 - e^{w_0} = 0$$

which implies $aw_0^2 - 2aw_0 = 0$. It follows that we must have either $w_0 = 0$ or $w_0 = 2$; i.e., either $0 = e^{w_0} = e^0 = 1$, or $Re(w_0) \geq 1$ and this is a contradiction. Therefore, for $R = 1$, the equation $aw^2 - e^w = 0$ $(|a| > e)$ has exactly 2 simple roots with positive real part located in $|w| < 1$.

Consequently, the equation $z^2e^z - \mu = 0$ has 2 solutions z satisfying $|z| < R$, where $0 \neq |\mu| < \frac{R^2}{e^R}$. In case when $R = 1$, these solutions are 2 simple roots with negative real part located in $|z| < 1$.

Let z_c be critical points in $|z| < R$. Using $ze^z = \frac{\mu}{z}$, we get

$$\begin{aligned} |f_\mu(z_c)| &\leq |e^{z_c} + \frac{\mu}{z_c}| \leq |e^{z_c}| + \frac{|\mu|}{|z_c|} \leq e^{|z_c|} + \frac{|\mu|}{|z_c|} \\ &\leq e^{|z_c|} + |z_c|e^{|z_c|} \leq (1 + |z_c|)e^{|z_c|} \\ &\leq (1 + R)e^R \end{aligned}$$

For some fix R, the quantity $(1 + R)e^R$ is bounded. It gives that the critical values of $f_\mu(z)$ are bounded.

On real axis, it is easily seen that the equation $x^2e^x - \mu = 0$ has finite number of zeros. It follows that $f_\mu(z)$ has finite critical values on real axis and all these critical values are bounded.

On imaginary axis, It is seen that $-y^2e^{iy} - \mu = 0$ which implies

$$y^2 \cos y = -\mu \quad \text{and} \quad y^2 \sin y = 0.$$

Second equation gives $y = (-1)^n n\pi$, where n is non-zero integer. Using first equation, the possible critical points on imaginary axis are $y = (-1)^n n\pi$, where n is non-zero odd integer. Consequently, the critical values $e^{(-1)^n n\pi} + \frac{\mu}{(-1)^n n\pi}$ are infinite in number. Since

$$|f_\mu((-1)^n n\pi i)| \leq 1 + \frac{|\mu|}{|n\pi|} < M.$$

It shows that infinitely many critical values of $f_\mu(z)$ on imaginary axis are bounded.

Hence, the function $f_\mu \in \mathcal{H}$ possesses infinitely many critical values and all these critical values of $f_\mu(z)$ are bounded.

Since $\underline{\lim}_{r\to\infty} \frac{n(r,\infty,f_\mu)}{\log r} < +\infty$, and Nevanlinna order ρ of the function $f_\mu(z)$ is 1, by Theorem 1, the function $f_\mu \in \mathcal{H}$ has at most two finite asymptotic values.

Thus, this proves that the function $f_\mu \in \mathcal{H}$ has infinitely many singular values and all these singular values of the function $f_\mu \in \mathcal{H}$ are bounded. □

3 Conclusion

It is concluded that the singular values of the one parameter family of singular perturbed exponential map has infinitely many bounded singular values.

Competing Interests

The author declares that no competing interests exist.

References

[1] Devaney R. L. Josić K. Shapiro Y. Singular Perturbations of Quadratic Maps, Int. J. Bifurcation and Chaos. 2004;14(1):161-169.

[2] Devaney R. L. Se^x: Dynamics, topology, and bifurcations of complex exponentials, Topology and its Applications. 2001;110(2):133-161.

[3] Rempe L. Dynamics of Exponential Maps, Christian-Albrechts-Universität Kiel, Germany, Doctoral thesis, 2003

[4] Argyris J. Karakasidis T. E. Andreadis I. On the Julia sets of a noise-perturbed Mandelbrot map, Chaos, Solitons & Fractals. 2002;13(2):245-252.

[5] Hayman W. K. Meromorphic functions, Clarendon Press; 1964.

[6] Avetision A. A. Arakelion N. U. Gonchar A. A. Asymptotic properties of meromorphic functions, Izvestiya Akademii Nauk Asmyamskay SSR Seriya Mathematika. 1989;24(3):207-222.

[7] Bergweiler W. Newton's method and a class of meromorphic functions without wandering domains, Ergodic Theory and Dynamical Systems. 1993;13(2):231-247.

[8] Sajid M. Kapoor G. P. Dynamics of a Family of Non-critically Finite Even Transcendental Meromorphic Functions, Regular and Chaotic Dynamics. 2004;9(2):143-162.

[9] Sajid M. Some Dynamical Properties of the Cosine-Meromorphic Family, Int. J. of Math. Sci. & Engg. Appls. 2014;8(1):91-99.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=487&id=6&aid=4232