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Hyers-Ulam-Rassias Stability by Fixed Point Technique for Half-linear Differential Equations with Unbounded Delay

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Abstract

In this paper we use the fixed point technique to establish the Hyers-Ulam-Rassias stability for half-linear differential equations with unbounded delay. Some illustrative examples are given.

Keywords: Hyers-Ulam-Rassias stability; fixed point; half-linear differential equation.

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1 Introduction

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940 (see [1]). In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [2] gave a partial solution to Ulam's problem. In 1978, Rassias [3] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians (see [4-11]). A generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, \; y(t), \; y'(t), ..., \; y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

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$$\left|F(t,\ y(t),\ y'(t),...,\ y^{(n)}(t))\right| \le \varepsilon$$

there exists a solution y_0 of the differential equation such that $|y(t)-y_0(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0.$

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [12,13]). Thereafter, Alsina and Ger published their paper [14], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \le \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \le 3\varepsilon$ for all $t \in (a, \infty)$. Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [15,16]). The results given in [17,18,19] have been generalized by Cimpean and Popa [20] and by Popa and Rus [21,22] for the linear differential equations of nth order with constant coefficients. In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order (see [23-27]). The Hyers-Ulam stability for various differential equations has been established by the author (e.g. see [28,29]). Jung and Brzdek [30] proved a similar type of stability for the delay linear equation with constant coefficient. The basic idea of proving is based on using the known general solution of $y' = \lambda y(t-\tau)$, where $\lambda > 0$ and $\tau > 0$ are real constants, where is given by $y(t) = \alpha \sum_{n=-1}^{[t/\tau]} \frac{\lambda^{n+1}(t-n\tau)^{n+1}}{(n+1)!}$. It should be noted that the equation (1.1) considered in this paper, is completely different than that investigated by Jung and Brzdek [30]. Burton [31] has used fixed point theory to establish Liapunov stability for functional differential equations.

Here, we use the fixed point approach to establish the Hyers-Ulam-Rassias stability for the half-linear differential equation with unbounded delay

$$y'(t) = -a(t)y(t) + b(t)g(y(t - r(t))). \tag{1.1}$$

We also investigate the half-linear differential equation

$$y'(t) = -a(t)y(t) + b(t)g(y(t - r(t))) + h(t). \tag{1.2}$$

Here, we assume that $a(t), b(t)$ and $r(t)$ are continuous, and that

$$\int_0^t |a(s)|ds \to \infty \ \text{ as } \ t \to \infty, \tag{1.3}$$

$$\int_0^t e^{-\int_s^t a(u)du} |b(s)|ds \le \alpha < 1, \ \ t \ge 0, \tag{1.4}$$

$$r(t) \ge 0, \ \ t - r(t) \to \infty \ \text{ as } t \to \infty, \tag{1.5}$$

Assume that there is a number $L > 0$ so that $|x|, |y| \le L$, implies that

$$g(0)=0 \text{ and } |g(x)-g(y)| \le |x-y|. \tag{1.6}$$

We also suppose that $h(t):[0,\infty)\to R$ with

$$\int_0^t e^{-\int_s^t a(u)du} |h(s)|ds \le \beta, \;\; t \ge 0. \tag{1.7}$$

Let $R^+ = [0,\infty)$ and $S = \{\phi : R \to R \mid \|\phi\| \le L, \;\; \phi(t)=\psi(t) \;\text{ if } t \le 0, \;\; \phi \in C \;\}$.

2 Preliminaries

Definition 2.1 We say that equation (1.2) (or (1.1) with $h(t)\equiv 0$) has the Hyers-Ulam-Rassias (HUR) stability with respect to φ if there exists a positive constant $k>0$ with the following property:

For each $y(t)\in C^1(R^+)$, if

$$| y'(t)+a(t)y(t)-b(t)g(y(t-r(t)))-h(t) | \le \varphi(t), \tag{2.1}$$

then there exists some $y_0(t)$ of the equation (1.2) such that $| y(t)-y_0(t) | \le k\,\varphi(t)$.

Theorem 2.1 the contraction mapping principle

Let $(S, \;\rho)$ be a complete metric space and let $P : S \to S$. If there is a constant $\alpha<1$ such that for each pair $\phi_1,\phi_2 \in S$ we have $\rho(P\phi_1,P\phi_2) \le \alpha\rho(\phi_1,\phi_2)$ then there is one and only one point $\phi \in S$ with $P\phi=\phi$.

3 Main Results on Hyers-Ulam-Rassias Stability

Theorem 3.1 Suppose that $y(t)\in C^1(R^+)$ satisfies the inequality (2.1) (with $h(t)\equiv 0$) with small continuous initial function $\psi:(-\infty, \;0]\to R$. Let $\varphi(t): [0,\infty)\to(0,\infty)$ be a continuous function such that

$$\int_0^t \varphi(s)e^{-\int_s^t a(u)du} ds \le C\,\varphi(t), \quad \forall t \ge 0. \tag{3.1}$$

If (1.3)-(1.6) hold then the solution of (1.1) with initial continuous function is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let $\psi:(-\infty, \;0]\to R$ be a continuous initial function with $\|\psi(t)\|<\delta$. Define $S = \{\phi : R \to R \mid \|\phi\| \le L, \;\; \phi(t)=\psi(t) \;\text{ if }\; t \le 0, \;\; \phi \in C \;\}$, where $\|\cdot\|$ is the supremum metric. Then $(S,\|\cdot\|)$ is a complete metric space. Now suppose that (1.3) holds. For L and α we find $\delta>0$ so that $\delta K+\alpha L \le L$, where

$$K = \sup_{t\ge 0}\{e^{-\int_0^t a(s)ds}\}.$$

Use the variation of parameters formula to write (1.1) as

$$y(t) = \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(s)ds} b(s)g(\phi(s-r(s)))ds.$$

Define $P : S \to S$ by

$$\left(P\phi\right)(t) = \psi(t) \text{ if } t \le 0,$$

and for $t \ge 0$ let

$$\left(P\phi\right)(t) = \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} b(s)g(\phi(s-r(s)))ds. \tag{3.2}$$

It is clear that for $\phi \in S, P\phi$ is continuous. Let $\phi(t) \in S$ with $\|\phi\| \le L$, for some positive constant L. Let ψ be a small given continuous initial function with $\|\psi\| \le \delta, \ \delta > 0$. Then using (1.3), (1.4) in the definition of $\left(P\phi\right)(t)$, we have

$$\begin{aligned} \|P\phi\| &\le \delta K + \int_0^t e^{-\int_s^t a(u)du} |b(s)||g(\phi(s-r(s)))|ds \\ &\le \delta K + \int_0^t e^{-\int_s^t a(u)du} |b(s)| L ds \le \delta K + \alpha L \end{aligned} \tag{3.3}$$

which implies that $\|P\phi\| \le L$. Thus (3.3) implies that $\left(P\phi\right)(t)$ is bounded.

To see that P is a contraction under the supremum metric, let $\phi, \eta \in S$. Then

$$\begin{aligned} \|\left(P\phi\right)(t) - (P\eta)(t)\| &\le \int_0^t e^{-\int_s^t a(u)du} |b(s)||g(\phi(s-r(s))) - g(\eta(s-r(s)))|ds \\ &\le \int_0^t e^{-\int_s^t a(u)du} |b(s)|\|\phi-\eta\|ds \le \alpha\|\phi-\eta\|, \quad \text{with } \alpha < 1. \end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point, say, y_0 in S which solves (1.1) and is bounded.

Next we show that the solution y_0 is stable in Hyers-Ulam-Rassias. Suppose we have the inequality

$$-\varphi(t) \le y'(t) + a(t)y(t) - b(t)g(y(t-r(t))) \le \varphi(t). \tag{3.4}$$

Multiplying the inequality (3.4) by $e^{\int_0^t a(u)du}$, we obtain

$$-\varphi(t)e^{\int_0^t a(s)ds} \le e^{\int_0^t a(s)ds} y'(t) + a(t)y(t)e^{\int_0^t a(s)ds} - b(t)g(y(t-r(t)))e^{\int_0^t a(s)ds}$$
$$\le \varphi(t)e^{\int_0^t a(s)ds}.$$

Or, equivalently, we have

$$-\varphi(t)e^{\int_0^t a(s)ds} \le \left(e^{\int_0^t a(s)ds} y(t)\right)' - b(t)g(y(t-r(t)))e^{\int_0^t a(s)ds}$$
$$\le \varphi(t)e^{\int_0^t a(s)ds}. \tag{3.5}$$

Integrating (3.5) from 0 to t, and then multiplying the obtained inequality by $e^{-\int_0^t a(u)du}$, we obtain

$$\left| y(t) - \psi(0)e^{-\int_0^t a(s)ds} - \int_0^t e^{-\int_s^t a(u)du} b(s)g(y(s-r(s))) \right| \le \int_0^t \varphi(s)e^{-\int_s^t a(u)du} ds \le C\varphi.$$

Using (3.2) we infer that $\|Py - y\| \le C\varphi$. To show that y_0 is stable we estimate the difference

$$\|y(t) - y_0(t)\| \le \|Py - y\| + \|Py - Py_0\| \le C\varphi + \alpha\|y - y_0\|.$$

Thus

$$\|y(t) - y_0(t)\| \le \frac{C\varphi}{1-\alpha}$$

which means that (2.1) (with $h(t) \equiv 0$) holds true for all $t \ge 0$.

Example 3.1 Consider the delay half-linear differential equation

$$y'(t) + a(t)y(t) = b(t)\sin(y(t-r(t))), \tag{3.6}$$

where $a(t) = \frac{1}{1+t}$, $b(t) = \frac{1}{(1+t)^3}$ and $r(t) = 0.05t$.

We estimate the integrals

$$\int_0^t |a(s)| \; ds = \int_0^t \frac{1}{1+s} ds = \ln(1+t) \to \infty \text{ as } t \to \infty,$$

and

$$\int_0^t e^{-\int_s^t a(u)du} |b(s)| \; ds = \int_0^t e^{-\int_s^t \frac{1}{1+u}du} \frac{1}{(1+s)^3} ds = \int_0^t e^{-\ln\left(\frac{1+t}{1+s}\right)} \frac{1}{(1+s)^3} ds$$

$$= \frac{1}{1+t}\int_0^t \frac{1}{(1+s)^2} ds = \frac{t}{(1+t)^2} \le \frac{1}{4} \le \alpha < 1, \quad t \ge 0.$$

If we set $\varphi(t) = e^t$, then we have

$$\int_0^t \varphi(s) e^{-\int_s^t a(u)du} ds = \int_0^t e^s e^{-\int_s^t \frac{1}{1+u}du} ds = \frac{te^t}{1+t} \le C\,\varphi(t), \;\; with\; C = 1,\; \forall t \ge 0.$$

Now since $g(y(t - r(t))) = \sin(y(0.95t))$, $|g(x) - g(y)| = |\sin x - \sin y| \le |x - y|$. The conditions of Theorem 3.1 are satisfied, hence the Eq. (3.6) is HUR stable for $t \ge 0$.

We now state the following corollary.

Corollary 3.1 Assume that $y(t) \in C^1(R^+)$ satisfies the inequality (3.4) with small continuous initial function $\psi : (-\infty, \; 0] \to R$, and that

$$\int_0^t e^{-\int_s^t a(u)du} ds \le C, \quad \forall t \ge 0. \tag{3.7}$$

If (1.3)-(1.6) hold then the solution of (1.1) with initial continuous function is stable in the sense of Hyers-Ulam.

Proof. It is enough to use Theorem 3.1, with $\varphi = \varepsilon$.

Remark 3.1 Suppose that $y(t) \in C^1(R^+)$ satisfies the inequality (2.1) (with $h(t) \equiv 0, a(t) \equiv 0$) with small continuous initial function $\psi : (-\infty, \; 0] \to R$. Let $\varphi(t) : [0, \infty) \to (0, \infty)$ be a continuous function such that

$$\int_0^t \varphi(s) ds \le C\,\varphi(t), \quad \forall t \ge 0. \tag{3.8}$$

and assume that

$$\int_0^t |b(s)| ds \le \alpha < 1, \; \forall t \ge 0.$$

If (1.5), (1.6) hold then the solution of (1.1) with initial continuous function is stable in the sense of Hyers-Ulam-Rassias.

Theorem 3.2 Suppose that $y(t) \in C^1(R^+)$ satisfies the inequality (2.1) with small continuous initial function $\psi : (-\infty,\ 0] \to R$. Let $\varphi(t) : [0,\infty) \to (0,\infty)$ be a continuous function such that

$$\int_0^t \varphi(s) e^{-\int_s^t a(u)du} ds \le C\,\varphi(t), \quad \forall t \ge 0. \tag{3.9}$$

If (1.3)-(1.7) hold then the solution of (1.2) with initial continuous function is stable in the sense of Hyers-Ulam-Rassias

Proof. First suppose that (1.3) holds. Then we can find $K > 0$ so that

$$K = \sup_{t \ge 0}\{e^{-\int_0^t a(s)ds}\}.$$

Let $\psi : (-\infty,\ 0] \to R$ be a continuous initial function with $\|\psi(t)\| < A_1,$ and determine $A_2 > 0$ with

$$KA_1 + \alpha A_2 + \beta = A_2.$$

Let $S = \{\phi : R \to R \mid \|\phi\| \le A_2,\ \ \phi(t) = \psi(t)\ \text{ if } t \le 0,\ \ \phi \in C\ \}$, where $\|\cdot\|$ is the supremum metric.

Then $(S, \|\cdot\|)$ is a complete metric space.

Using the variation of parameters formula we write (1.2) as

$$y(t) = \psi(0) e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(s)ds} b(s) g(\ \phi(s - r(s)))ds + \int_0^t e^{-\int_s^t a(s)ds} h(s)ds.$$

Define $P : S \to S$ by $(P\phi)(t) = \psi(t)$ if $t \le 0,$
and for $t \ge 0$ let

$$(P\phi)(t) = \psi(0) e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(s)ds} b(s) g(\ \phi(s - r(s)))ds + \int_0^t e^{-\int_s^t a(s)ds} h(s)ds. \tag{3.10}$$

It is clear that for $\phi \in S$, $P\phi$ is continuous. Assume that $\phi(t) \in S$ with $\|\phi\| \le A_2$, and that ψ is a small given continuous initial function with $\|\psi\| \le A_1,\ A_1 > 0$.

Using (1.3), (1.4) and (1.7) in the definition of $(P\phi)(t)$ and applying (1.6), we have

$$\begin{aligned}\|P\phi\| &\le A_1 K + \int_0^t e^{-\int_s^t a(u)du} |b(s)|\ \ \|\phi\|\ \ ds + \int_0^t e^{-\int_s^t a(s)ds} |h(s)|\ \ ds \\ &\le A_1 K + \alpha A_2 + \beta = A_2\end{aligned}$$

which implies that $(P\phi)(t)$ is bounded.

To see that P is a contraction under the supremum metric, let $\phi,\eta\in S$. Then

$$\|(P\phi)(t)-(P\eta)(t)\|\le\int_0^t e^{-\int_s^t a(u)du}|b(s)||g(\phi(s-r(s)))-g(\eta(s-r(s)))|ds$$
$$\le\int_0^t e^{-\int_s^t a(u)du}|b(s)|\|\phi-\eta\|ds\le\alpha\|\phi-\eta\|,\quad \text{with } \alpha<1.$$

Thus, by the contraction mapping principle, P has a unique fixed point, say, y_0 in S which solves (1.2) and is bounded.

Next we show that the solution y_0 is HUR stable. From the inequality (2.2) we get

$$-\varphi(t)\le y'(t)+a(t)y(t)-b(t)g(y(t-r(t)))-h(t)\le\varphi(t) \tag{3.11}$$

Multiplying the inequality (3.11) by $e^{\int_0^t a(u)du}$, we obtain

$$\left|\left(e^{\int_0^t a(s)ds}y(t)\right)'-b(t)g(y(t-r(t)))e^{\int_0^t a(s)ds}-\int_0^t e^{-\int_s^t a(s)ds}h(s)ds\right|\le\varphi(t)e^{\int_0^t a(s)ds} \tag{3.12}$$

Integrating the inequality (3.12) from 0 to t, and then multiplying the obtained inequality by $e^{-\int_0^t a(u)du}$, we obtain

$$\left|y(t)-\psi(0)e^{-\int_0^t a(s)ds}-\int_0^t e^{-\int_s^t a(u)du}b(s)g(y(s-r(s)))-\int_0^t e^{-\int_s^t a(s)ds}h(s)ds\right|$$
$$\le\int_0^t\varphi(s)e^{-\int_s^t a(u)du}ds\le C\varphi$$

It follows from (2.2) that $\|Py-y\|\le C\varphi$. We estimate the difference

$$\|y(t)-y_0(t)\|\le\|Py-y\|+\|Py-Py_0\|\le C\varphi+\alpha\|y-y_0\|.$$

Thus

$$\|y(t)-y_0(t)\|\le\frac{C\varphi}{1-\alpha},$$

which means that the inequality (2.2) holds true for all $t\ge 0$.

Example 3.2 Consider the delay half-linear differential equation

$$y'(t)+a(t)y(t)=b(t)\sin(y(t-r(t)))+h(t), \tag{3.13}$$

where $a(t)=\frac{1}{1+t}$, $b(t)=\frac{1}{(1+t)^3}$, $r(t)=0.90t$ and $h(t)=\frac{e^{-t}}{1+t}$.

One can similarly, as in Example 3.1 establish the validity of conditions (1.3)-(1.6). So, to show that (3.13) is stable in the sense of Hyers-Ulam-Rassias, it remains to estimate the integral

$$\int_0^t e^{-\int_s^t a(s)ds} |h(s)| \ ds = \int_0^t e^{-\int_s^t \frac{1}{1+u}du} \frac{e^{-s}}{1+s} ds = \frac{1-e^{-t}}{1+t} \le \frac{1}{3}, \quad \forall t \ge 0.$$

Hence the conditions of Theorem (3.2) are satisfied.

Corollary 3.2 Assume that $y(t)\in C^1(R^+)$ satisfies the inequality (2.1) with small continuous initial function $\psi:(-\infty,\ 0]\to R$, and that

$$\int_0^t e^{-\int_s^t a(u)du} ds \le C, \quad \forall t \ge 0. \tag{3.14}$$

If (1.3)-(1.7) hold then the solution of (1.2) with initial continuous function is stable in the sense of Hyers-Ulam.

Proof. It is enough to use Theorem 3.2, with $\varphi=\varepsilon$.

Remark 3.2 Suppose that $y(t)\in C^1(R^+)$ satisfies the inequality (2.1) (with $a(t)\equiv 0$) with small continuous initial function $\psi:(-\infty,\ 0]\to R$. Let $\varphi(t):[0,\infty)\to(0,\infty)$ be a continuous function such that

$$\int_0^t \varphi(s)ds \le C\varphi(t), \quad \forall t \ge 0.$$

and assume that

$$\int_0^t |b(s)| \ ds \le \alpha < 1.$$

If (1.5)-(1.7) hold then the solution of (1.2) with initial continuous function is stable in the sense of Hyers-Ulam-Rassias.

4 Conclusion

In this paper we have obtained integral criteria for Hyers-Ulam-Rassias stability of half-linear differential equations with unbounded delay. Some illustrative examples are given.

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Competing Interests

Author has declared that no competing interests exist.

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