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Useful Formulas for One-dimensional Differential Transform

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this work, we introduced new useful formulas for one-dimensional differential transform and applied the differential transform method to selected linear ordinary differential equations. This study showed that this method is powerful and efficient in finding series solutions for linear differential equations and capable of reducing the size of calculations comparing with other methods.

Keywords: Differential transform; Taylor series method; differential equations.

1. INTRODUCTION

The idea of differential transform presented by G. E. Pukhov [1] in 1982 and the concept of differential transform proposed first by Zhou [2] in 1986 when applied to solve linear and nonlinear initial value problems in electric circuit analysis. The (Pukhov-Zhou) differential transform method

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presented as a new method based on Taylor series [3-14] and consider as a semi-analytical technique uses the Taylor series to construct the solutions of differential equations in the form of a power series. This method represents an iterative procedure for obtaining analytic series solutions of differential equations and useful for obtaining exact and approximate solutions of linear ordinary differential equations [4-6] and system of linear ordinary differential equations [8-9]. The main aims in this work are to introduce new useful algorithms for one-dimensional differential transform and applied the differential transform method to selected linear ordinary differential equations.

2. THE ONE-DIMENSIONAL DIFFERENTIAL TRANSFORM

In this section, we introduce the concept of one-dimensional differential transform and review some basic fundamental theorems [4-13]. To do that, we assume that $f(x)$ be a $C^\infty(I)$ function in an open interval of R, and x_0 be any point of I. Then the Taylor series of $f(x)$ about x_0 can be represented by

$$f(x)=\sum_{k=0}^{\infty}\left[\frac{f^{(k)}(x)}{k!}\right]_{x=x_0}(x-x_0)^k \tag{2.1}$$

Note that, throughout this work, we assume that $f(x)$ is an analytic function at every point of I.

Definition: Let $f(x)$ be an analytic function about x_0, then the k^{th} order differential transform of $f(x)$ is defined as

$$D_T\{f(x)\}:=\left[\frac{f^{(k)}(x)}{k!}\right]_{x=x_0}. \tag{2.2}$$

In definition (2.1), $D_T\{f(x)\}$ represents the one-dimensional differential transform of $f(x)$ about x_0 and it usually denoted by $F(k)$ and throughout our work, we will take $x_0=0$, which reduces the definition (2.1) to

$$D_T\{f(x)\}=F(k):=\frac{f^{(k)}(0)}{k!}. \tag{2.3}$$

Note that, the inverse differential transform of $F(k)$ denoted by $D_T^{-1}\{F(k)\}$ is defined as

$$D_T^{-1}\{F(k)\}=f(x):=\sum_{k=0}^{\infty}F(k)x^k \tag{2.4}$$

For example $D_T\{e^{\alpha x}\}=\alpha^k/k!$, $D_T\{\cos\alpha x\}=(\alpha^k/k!)\cos(k\pi/2)$, respectively, where, α is a constant.

Theorem (1): Let $f(x)$ and $g(x)$ be analytic functions, with differential transforms $F(k)$ and $G(k)$ respectively, then

$$D_T\{\alpha f(x)+\beta g(x)\}=\alpha D_T\{f(x)\}+\beta D_T\{g(x)\}, \tag{2.5}$$

$$D_T^{-1}\{\alpha F(k)+\beta G(k)\}=\alpha D_T^{-1}\{F(k)\}+\beta D_T^{-1}\{G(k)\}, \tag{2.6}$$

where α and β are constants. The proof of the linearity property follows immediately from the definitions (1) and (2).

Theorem (2): Let $f(x)$ be an analytic function, with $D_T\{f(x)\}=F(k)$, then

$$D_T\left\{\frac{d^n f(x)}{dx^n}\right\}=\frac{(k+n)!}{k!}F(k+n) \tag{2.7}$$

In fact,

$$D_T\left\{\frac{d^n f(x)}{dx^n}\right\}=D_T\left\{\sum_{k=0}^{\infty}\frac{f^{(k+n)}(0)}{k!}x^k\right\}=D_T\left\{\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{f^{(k+n)}(0)}{(k+n)!}x^k\right\}$$

$$=\frac{(k+n)!}{k!}\frac{f^{(k+n)}(0)}{(k+n)!}=\frac{(k+n)!}{k!}F(k+n). \tag{2.8}$$

Theorem (3): Let $F(k)$ be the differential transform of $f(x)=x^m$, then $F(k)=\delta_{km}$, where, δ_{km} is the Kronecker delta. The proof of this theorem follows immediately from identity $x^m \equiv \sum_{k=0}^{\infty}\delta_{km}x^k$.

Theorem (4): Let $f(x)$ and $g(x)$ be analytic functions and

$$f(x)=\int_0^x g(t)\,dt, \tag{2.9}$$

then the differential transform of the function $f(x)$ is given by

$$D_T\left\{f(x)\right\}=\frac{G(k-1)}{k}, \tag{2.10}$$

where $G(k)$ is the differential transform of $f(x)$. Because $f'(x)=g(x)$, we get from (2.8)

$$\frac{g^{(k)}(0)}{k!}=\frac{(k+1)!}{k!}\frac{f^{(k+1)}(0)}{(k+1)!}=(k+1)\frac{f^{(k+1)}(0)}{(k+1)!}.$$

This can be written in the form

$$\frac{f^{(k)}(0)}{k!}=\frac{1}{k}\frac{g^{(k-1)}(0)}{(k-1)!},$$

which his yields the formula (2.10).

Theorem (5): Let $f_1(x)$ and $f_2(x)$ be analytic functions, with $f(x)=f_1(x)\cdot f_2(x)$, then

$$D_T\left\{f(x)\right\}=\sum_{n=0}^{k}F_1(n)F_2(k-n), \tag{2.11}$$

where $F_1(k)$ and $F_2(k)$ are the differential transforms of the functions $f_1(x)$ and $f_2(x)$ respectively. To prove (2.11), we first write

$$f_1(x)\,f_2(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{f_1^{(n)}(0)}{n!}\frac{f_2^{(m)}(0)}{m!}\frac{x^{n+m}}{n!m!}=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f_1^{(n)}(0)}{n!}\frac{f_2^{(k-n)}(0)}{(k-n)!}\frac{x^k}{n!(k-n)!}$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{f_1^{(n)}(0)}{n!}\frac{f_2^{(k-n)}(0)}{(k-n)!}\frac{x^n x^{k-n}}{n!(k-n)!}$$

Next, applying the differential transform on both sides of the above equation yields the formula (2.11).

3. NEW FORMULAS

In this section, we introduce new basic formulas for the one-dimensional differential transform.

Theorem (6): Let $f(x)$, be an analytic function, with $D_T\left\{f(x)\right\}=F(k)$, then

$$D_T\left\{x^m f^{(n)}(x)\right\}=\sum_{i=0}^{k}\delta_{im}\frac{(k+n-i)!}{(k-i)!}F(k+n-i) \tag{3.1}$$

To prove the formula (3.1), we make use of (8) and $D_T\left\{x^m\right\}=\delta_{km}$. This enables us to write

$$F_1(k)=D_T\left\{x^m\right\}=\delta_{km} \qquad F_2(k)=D_T\left\{f^{(n)}(x)\right\}=\frac{(k+n)!}{k!}F(k+n). \tag{3.2}$$

Then, the formula (2.11) leads to

$$D_T\left\{x^m f^{(n)}(x)\right\} = \sum_{i=0}^{k} F_1(i) F_2(k-i) \qquad (3.3)$$

Finally, if we substitute (3.2) into (3.3), we can establish the formula (3.1). Note that, for $m=0$, the formula (3.1) reduces to the formula (2.7).

Corollary (1): Let $f(x)$, be an analytic function, with $D_T\left\{f(x)\right\} = F(k)$, then

$$D_T\left\{x^n f^{(n)}(x)\right\} = \prod_{i=0}^{n-1}(k-i)F(k). \qquad (3.4)$$

Note that, making use of (3.1) and the identity

$$k! = \frac{(k-n+1)!}{(k-n+1)(k-n)(k-n-1)(k-n-2)....(k-1)}, \qquad (3.5)$$

we can easily establish the formula (3.4).

Corollary (2): Let $f(x)$ be an analytic function, with $D_T\left\{f(x)\right\} = F(k)$, then

$$D_T\left\{e^{\alpha x} f^{(n)}(x)\right\} = \sum_{i=0}^{k} \frac{\alpha^i}{i!} \frac{(k+n-i)!}{(k-i)!} F(k+n-i) \qquad (3.6)$$

To prove (3.6), we make use of (2.2) and (2.7) to write

$$F_1(k) = D_T\left\{e^{\alpha t}\right\} = \frac{\alpha^k}{k!}, \qquad F_2(k) = D_T\left\{f^{(n)}(x)\right\} = \frac{(k+n)!}{k!} F(k+n). \qquad (3.7)$$

Then, the formula (2.11) leads to

$$D_T\left\{e^{\alpha x} f^{(n)}\right\} = \sum_{i=0}^{k} F_1(i) F_2(k-i) = \sum_{i=0}^{k} \frac{\alpha^i}{i!} \frac{(k+n-i)!}{(k-i)!} F(k+n-i) \qquad (3.8)$$

Theorem (7): Let $f(x)$ be an analytic function, with $D_T\left\{f(x)\right\} = F(k)$, then

$$D_T\left\{\left(xf^{(n)}(x)\right)'\right\} = \frac{(k+1)(k+n)!}{k!} F(k+n) \qquad (3.9)$$

We can establish the proof of the formula (3.8) by using (2.4), (6) and (2.11) as following

$$D_T\left\{\left(xf^{(n)}(x)\right)'\right\} = D_T\left\{xf^{(n+1)}(x)\right\} + D_T\left\{f^{(n)}(x)\right\} = \frac{(k+n)!}{(k-1)!} F(k+n) + \frac{(k+n)!}{k!} F(k+n)$$

$$= \frac{(k+n)!}{(k-1)!}\left[1+\frac{1}{k}\right] F(k+n) = \frac{(k+1)(k+n)!}{k!} F(k+n).$$

Note that, for $n=1$ the formula (3.9), reduce to the formula

$$D_T\left\{\left(xf'(x)\right)'\right\} = (k+1)^2 F(k+1) \qquad (3.10)$$

Theorem (8): Let $f(x)$ be an analytic function with $D_T\left\{f(x)\right\}=F(k)$, then

$$D_T\left\{\left(x^m f^{(n)}(x)\right)'\right\}=\frac{(k+1)(k+n-m+1)!}{(k-m+1)!}F\left(k+n-m+1\right) \tag{3.11}$$

We can establish the proof of the formula (3.11) by using (2.5) and (3.1) as

$$D_T\left\{\left(x^m f^{(n)}(x)\right)'\right\}=D_T\left\{x^m f^{(n+1)}(x)\right\}+mD_T\left\{x^{m-1} f^{(n)}(x)\right\}$$

$$=\frac{(k+n+1-m)!}{(k-m)!}F(k+n+1-m)+m\frac{(k+n-m+1)!}{(k-m+1)!}F(k+n-m+1)$$

Now by setting $h=m-1$, we can write

$$D_T\left\{\left(x^m f^{(n)}(x)\right)'\right\}=\frac{(k+n-h)!}{(k-h-1)!}F(k+n-h)+(h+1)\frac{(k+n-h)!}{(k-h)!}F(k+n-h)$$

$$=\frac{(k+n-h)!}{(k-h-1)!}F(k+n-h)+\frac{(h+1)}{(k-h)}\frac{(k+n-h)!}{(k-h-1)!}F(k+n-h)$$

$$=\left(1+\frac{(h+1)}{(k-h)}\right)\frac{(k+n-h)!}{(k-h-1)!}F(k+n-h)$$

$$=\left(\frac{(k+1)}{(k-h)}\right)\frac{(k+n-h)!}{(k-h-1)!}F(k+n-h)=\frac{(k+1)(k+n-h)!}{(k-h)!}F(k+n-h).$$

This yields the formula (3.12).

4. NUMERICAL EXAMPLES

In this section, we apply the differential transform method to selected linear differential equations and compare our results with the results obtained by the Taylor series method.

Example (1)

Consider the initial value problem

$$y''-2xy'+2ny=0, \tag{4.1}$$

$$y(0)=c_0, \quad y'(0)=c_1, \tag{4.2}$$

The differential equation (4.1) is known as Hermite equation, where n is usually a non-negative integer. To solve the initial value problem (4.1-2), we apply the differential transform to both sides of (4.1). This gives

$$D_T\left\{y''\right\}-2D_T\left\{xy'\right\}+2nD_T\left\{y\right\}=0. \tag{4.3}$$

Next, making use of the formulas (2.7) and (3.1) enables us to find

$$(k+1)(k+2)Y(k+2)-2kY(k)+2nY(k)=0. \tag{4.4}$$

This leads to the iteration formula

$$Y(0)=c_0,\qquad Y(1)=c_1,\qquad Y(k+2)=\frac{2(k-n)}{(k+1)(k+2)}Y(k),\ k=0,1,2\dots \tag{4.5}$$

Now, the iteration formula (4.5) gives

$$Y(2)=\frac{-2n}{2!}c_0,\ Y(3)=\frac{-2(n-1)}{3!}c_1,\qquad Y(4)=\frac{2^2 n(n-2)}{4!}c_0,$$

$$Y(5)=\frac{2^2(n-1)(n-3)}{5!}c_1,\qquad Y(6)=\frac{-2^3 n(n-2)(n-4)}{6!}c_0. \tag{4.6}$$

Therefore, this leads to the series solution

$$y(x)=c_0\left(1-\frac{2n}{2!}x^2+\frac{2^2 n(n-2)}{4!}x^4-\frac{2^3 n(n-2)(n-4)}{6!}x^6\dots\right)+c_1\left(x-\frac{2(n-1)}{3!}x^3+\frac{2^2(n-1)(n-3)}{5!}x^5+\dots\right). \tag{4.7}$$

This is the Hermite polynomial given in ref. [15].

Example (2)

Consider the initial value problem

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0, \tag{4.8}$$

$$y(0)=c_0,\qquad y'(0)=c_1, \tag{4.9}$$

The self-adjoint equation (4.8) is known as the Legendre equation, where, c_0 and c_1 are constant and n is an integer. To solve the initial value problem (4.8-9), we apply the differential transform to both sides of (4.8). This gives

$$D_T\left\{y''\right\}-D_T\left\{(x^2y')'\right\}+D_T\left\{n(n+1)y\right\}=0. \tag{4.10}$$

Next, making use of the formulas (2.7) and (3.11) enables us to find

$$(k+1)(k+2)Y(k+2)-k(k+1)Y(k)+n(n+1)Y(k)=0, \tag{4.11}$$

This leads to the iteration formula

$$Y(0)=c_0,\qquad Y(1)=c_1$$

$$Y(k+2)=\frac{1}{(k+1)(k+2)}\left[k(k+1)-n(n+1)\right]Y(k),\quad k=0,1,2\dots \tag{4.12}$$

Now, the iteration formula (4.12) gives

$$Y(2)=-\frac{n(n+1)}{2!}c_0,\ Y(3)=-\frac{(n-1)(n+2)}{3!}c_1,\qquad Y(4)=\frac{(n-2)n(n+1)(n+3)}{4!}c_0,$$

$$Y(5)=\frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1,\ Y(6)=-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0,$$

$$Y(7)=-\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1,\dots. \tag{4.13}$$

Therefore, this leads to the series solution

$$y(t)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\ldots\right]+$$
$$c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\right.$$
$$\left.\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\ldots\ldots\right] \tag{4.14}$$

This result is identical to the result obtained by the Taylor series procedure applied in ref. [16].

Example (3)

Consider the initial value problem

$$y'-2xy=0\ , \tag{4.15}$$

$$y(0)=c_0\,. \tag{4.16}$$

To solve the initial value problem (4.15-16), we make use of the formulas (7), (11) and (13). This gives

$$(k+1)Y(k+1)-2\sum_{i=0}^{k}\delta_{i1}Y(k-i)=0\,. \tag{4.17}$$

Now, we can write the iteration formula

$$Y(0)=c_0,\quad Y(k+1)=\frac{2}{k+1}\sum_{i=0}^{k}\delta_{i1}Y\left(k-i\right),\qquad k=0,1,2,\ldots \tag{4.18}$$

The iteration formula (4.18) for $k=0,1,2,\ldots$ gives

$$Y(0)=c_0,\qquad Y(1)=0,\qquad Y(2)=c_0,\qquad Y(3)=0,$$

$$Y(4)=\frac{c_0}{2!},\qquad Y(5)=0,\qquad Y(6)=\frac{c_0}{3!},\ldots.$$

Therefore, the close form solution can be easily in the form written as

$$y(x)=\sum_{k=0}^{\infty}Y[k]x^k=c_0\left(1+x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\frac{1}{4!}x^8\ldots\ldots\right)=c_0\sum_{k=0}^{\infty}\frac{1}{k!}x^{2k}\,.$$

This represents the Taylor series expansion of the exact solution $y(x)=c_0e^{x^2}$.

5. CONCLUSION

In this work, we introduced the concept of differential transform and studied some of their properties. As a contribution, we introduced and proved new formulas for the one-dimensional differential transform and applied the differential transform method to selected linear ordinary differential equations. This study showed that the differential transform method is powerful and efficient techniques in finding analytical solutions for linear differential equations. It also showed that this technique is capable of reducing the size of calculations comparing with the Taylor series method.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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