An Operational Matrix of Hermite Polynomials for Solving Nonlinear Fractional Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Abstract

In this paper, an effective technique known as the operational matrix method is utilised to solve nonlinear form of fractional differential equations (FDEs). An explicit effort is placed on the derivation of Hermite polynomials operational matrix with the Caputo sense. The main motivation behind this work is to convert a nonlinear type of FDE into a set of algebraic equations with the consideration of initial conditions. The problem is therefore simplified by these equations to be solved with the proposed method. In order to confirm the effectiveness of the proposed approach, numerical and analytical solutions for a number of nonlinear FDEs are presented. Due to the high simplicity of the proposed approach in practice, it can be comfortably implemented in various aspects of applied science domain.

1 Introduction

It is well-understood that many practical issues in the science domain such as physics, chemistry, engineering and diverse parts of science have been success- fully described by the fractional calculus

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phenomena, whereby efficient models are developed with mathematical tools [1]. Fractional differential equations (FDEs) have been recently gaining a significant attention in defining such a real-world application. It is a very difficult task to find an exact or approximate solution for FDEs. Therefore, a critical effort has been extensively placed on the solutions of FDEs. Several attempts have been performed to propose novel strategies for either exact or approximated solutions of FDEs [2].

Many of the existing studies for solving FDEs focus only on linear type of FDEs as it is relatively straightforward to be solved. An effective way of solving the FDEs is spectral methods presenting very accurate approximations. A number of recent work introduced operational matrices for Caputo-type fractional derivatives to be applied with spectral methods through different kinds of polynomials, subject to pre-given initial conditions. Typical examples are conducted with the polynomials of Chebyshev [3], shifted Chebyshev [4], Legendre [5], Genocchi [6] and Jacobi [1]. The fundamental operation of these studies is to simplify the complex structure of the FDEs to an algebraic system with the introduction of the relevant polynomials. Then, the exact solutions are obtained based upon the solution of the algebraic system.

On the other hand, approximate solutions for non-linear FDEs systems are carefully required to be established. In this respect, several implementations of operational matrix model have resulted in the high accuracy for non-linear FDEs. For example, an operational matrix of fractional derivatives for gener- alized Laugerre was constructed to propose an ecient solution for the system of non-linear FDEs [7]. In order to solve non-linear FDEs, this paper targets at derivation of a new operational matrix with Hermite polynomials with respect to Caputo-sense fractional derivative. The obtained operational matrix is applied with spectral Tau scheme to serve to approach the solution of FDEs. Algebraic equations are constructed by converting the FDEs into a set of integral forms with the introduction of the operational matrix, and extra algebraic equations extracted from the initial conditions. By solving the system of algebraic equations, we present a good approximate solutions for a number of numerical examples. The remainder of the paper is given in the fol- lowing. We start with summarizing the necessary preliminaries and notations related to the complete structure of the proposed idea. In the following section, the operational matrix for fractional derivative through Hermite polynomials is derived. Then, the derived operational matrix is applied for solving the non- linear FDEs. A set of representative examples with the exact or approximated solutions are introduced. The last section finally concludes the paper.

2 Preliminaries

We start by giving definitions of fractional derivatives and Hermite polynomials.

Definition 2.1. The Rieman-Liouville fractional integral with the order of v is given as:

$$J^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt \qquad (2.1)$$

$$J^0 f(x) = f(x). \qquad (2.2)$$

Definition 2.2. The Caputo derivative operator with the order v, subject to $v > 0$, is:

$$D^v f(x) = J^{m-v} D^m f(x) = \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{(m-v-1)} \frac{d^m}{dt^m} f(t) dt, \qquad (2.3)$$

$$m - 1 < v < m, m \in \mathbf{N} x > 0. \qquad (2.4)$$

and this relation is satisfied for the exponential functions:

$$D^v x^\beta == \begin{cases} 0, & \text{for } \beta \in N \text{and} \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & \text{for } \beta \in N \text{ and } \beta \geq \lceil v \rceil \end{cases}$$

Here, $\lfloor v \rfloor$ and $\lceil v \rceil$ indicate the functions of floor and ceiling in turn with $N = 1, 2, 3, ...$

The Caputo derivative operator satisfies linearity condition such as:

$$D^v(\lambda f(x) + \mu g(x)) = \lambda D^v f(x) + \mu D^v g(x), \tag{2.5}$$

where λ and μ take constant values.

We note that when $v \in N$ the definition of the Caputo derivative and usual differential operator definition are the same.

2.1 Properties of Hermite Polynomials

The Hermite polynomials defined on $(-\infty, \infty)$ can be given analytically as [8]:

$$H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k (2x)^{i-2k}}{k!(i-2k)!} \tag{2.6}$$

or we can define Hermite polynomials as: [8]

$$H_i(x) = (-1)^k e^{-x^2} \frac{d^i e^{-x^2}}{dx^i} \tag{2.7}$$

The series of Hermite polynomials exhibit an orthogonal system,

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) e^{(-x^2)} dx = \begin{cases} 0, & i \neq j \\ 2^i i! \sqrt{\pi}, & i = j \end{cases}$$

where e^{-x^2} is the weight function of Hermite polynomials. The recurrence relation is satisfied by Hermite polynomials:

$$H_{i+1}(x) = 2x H_i(x) + 2i H_{i+1}(x) \tag{2.8}$$

3 Materials and Methods

A square integrable function defined on $(-\infty, \infty)$, called $u(x)$, can be written by Hermite polynomials like

$$u(x) = \sum_{i=0}^{\infty} c_i H_i(x) \tag{3.1}$$

then the coefficients c_i is defined as

$$c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{\infty} u(x) H_i(x) w(x) dx, i = 0, 1, 2, ... \tag{3.2}$$

The truncated $N + 1$ Hermite polynomial series are

$$u(x) = \sum_{i=0}^{N} c_i H_i(x) = C\Psi(x) \tag{3.3}$$

the C represents a vector with the unknown coefficients, and $\Psi(x)$ is the Hermite vector to be given as

$$C = \begin{bmatrix} c_0 & c_1 & \ldots & c_N \end{bmatrix}$$

$$\Psi(x) = [H_0(x), H_1(x), ..., H_N(x)] \tag{3.4}$$

The derivative of vector $\Psi(x)$ is

$$\frac{d\Psi(x)}{dx} = \boldsymbol{D^{(1)}}\boldsymbol{\Psi(x)} \tag{3.5}$$

where $D^{(1)}$ contains the operational matrix with the size of $(N+1) \times (N+1)$.

If we repeat the derivative p time:

$$\boldsymbol{D^p}\boldsymbol{\Psi(x)} = \boldsymbol{D^{(p)}}\Psi(x) \tag{3.6}$$

Here, $D^{(p)}$ corresponds to the aforementioned matrix for the integer values of p.

Lemma 3.1. *Let $H_i(x)$ is the Hermite polynomial; then*

$$D^v H_i(x) = 0, i = 0, 1, 2, ..., \lceil v \rceil - 1, v \geq 0. \tag{3.7}$$

Theorem 3.2. *Let $\Psi_i(x)$ is the Hermite polynomial and if $v \geq 0$; then*

$$D^v \Psi(x) = D^{(v)} \Psi(x) \tag{3.8}$$

where $D^{(v)}$ illustrates the related operational matrix for Hermite polynomials with the same dimensions $(N+1) \times (N+1)$, that is

$$\mathbf{D}^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \Omega_v(i,0) & \Omega_v(i,1) & \Omega_v(i,2) & \ldots & \Omega_v(i,N) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \Omega_v(N,0) & \Omega_v(N,1) & \Omega_v(N,2) & \ldots & \Omega_v(N,N) \end{pmatrix}$$

where

$$\Omega_v(i,j) = \sum_{k=0}^{\lfloor \frac{i-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^{(k+r)} 2^{(i-2k+j-2r)} i! j! \frac{\Gamma(i-2k+j-2r+1)}{2}}{2(j-2r)! k! r! \Gamma(i-2k+1-v)}, j = 0, 1, ..., N \tag{3.9}$$

Proof. The analytic formula of Hermite Polynomials is given by Eq.(2.6) and by using Eq.(2.6) we obtain

$$D^v H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k 2^{i-2k} D^v(x^{i-2k})}{k!(i-2k)!} \tag{3.10}$$

From the definition of the Caputo derivative of exponential function we get the derivative of x^{i-2k}

$$D^v H_i(x) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k 2^{i-2k}(x^{i-2k-v})}{k! \Gamma(i-2k-v+1)}, i = \lceil v \rceil, \lceil v \rceil + 1, ..., N. \tag{3.11}$$

When approximating $x^{(i-2k-v)}$ by $N+1$ Hermite series as

$$x^{(i-2k-v)} = \sum_{j=0}^{N} c_j H_j(x) \tag{3.12}$$

from the definition of c_j in Eq.(3.2) and by taking into consideration Lemma 3.1 we get

$$D^v H_i(x) = \sum_{k=0}^{\lfloor \frac{i-\lceil v \rceil}{2} \rfloor} \frac{(-1)^k 2^{i-2k} \Gamma(i-2k+1) x^{i-2k+1}}{k!(i-2k)!\Gamma(i-2k-v+1)}, i = \lceil v \rceil, \lceil v \rceil + 1, ..., N. \tag{3.13}$$

If we employ Eqs.(3.10) -(3.13) we obtain

$$D^v H_i(x) = \sum_{j=0}^{N} \Psi_v(i,j) H_j(x) \tag{3.14}$$

When extracting $\Psi_v(i,j)$ from Eq.(3.14):

$$\Psi_v(i,j) = \sum_{k=0}^{\lfloor \frac{i-\lfloor v \rfloor}{2} \rfloor} \frac{1}{2^j j! \sqrt{\pi}} \sum_{r=0}^{\lfloor j/2 \rfloor} \frac{(-1)^{k+r} 2^{i-2k+j-2r} j! i! \Gamma(i-2k+j-2r+1/2)}{2(j-2r)!\Gamma(i-2k+1-v)k!r!} \tag{3.15}$$

$\square$

3.1 Non-linear Multi-Order FDEs

In this sub-section we mention how operational matrix is applied to the nonlinear FDEs with Tau method. After u_N, the extracted solution is composed by Hermite polynomials, and $N+1$ system of equations is solved for the approximate solutions.

Here nonlinear fractional differential equations are

$$D^v u(x) = F(x, u(x), D^{\beta_1} u(x) + D^{\beta_2} u(x) + ..., D^{\beta_k}), \tag{3.16}$$

with these initial conditions

$$u^{(i)}(0) = d_i, i = 0, 1, ..., m-1. \tag{3.17}$$

where F can be nonlinear with the conditions of $m-1 < v < m$ and $0 < \beta_1 < \beta_2 < ... < \beta_k < v$. The d_i values $d_i, i = (0, 1, ..., m-1)$ represents the initial conditions.

For the approximate solution of the initial value problem (3.16)-(3.17) we convert the $u(x)$, $D^v u(x)$ and $D^{\beta_j} u(x)$ into the equations respectively

$$u(x) = C^T \Psi(x), \tag{3.18}$$

$$D^v u(x) = C^T \mathbf{D}^{(v)} \Psi(x) = \mathbf{D}^{(\beta_j)} \Psi(x) \tag{3.19}$$

and

$$D^{\beta_j} \Psi(x) = C^T \mathbf{D}^{(\beta_j)} \Psi(x) \tag{3.20}$$

The equations (3.18)-(3.20) are substituted into Eq.(3.16) then an equation is reconstituted. Then, $N-m+1$ roots of Hermite polynomials is used instead of x in Eq.(3.16); in other words we collocate Eq.(3.16) in $N-m+1$ roots of Hermite polynomials. From the conditions we have m equations then we have $N+1$ equations at the end. By Newton iteration method we solve the system of equations.

4 Applications of Operational Matrix for Non-Linear FDEs

Examples of some equations are given below:

Example 4.1. *Our first example is the initial value problem [3]*

$$D^3 u(x) + D^{5/2} u(x) + u^2(x) = x^4, u(0) = 0, u'(0) = 0, u''(0) = 2 \tag{4.1}$$

The exact solution for this problem is

$$u(x) = x^2 \tag{4.2}$$

we apply the technique described in Section 5.1 for $N = 3$

$$u(x) = \sum_{i=1}^{3} a_i H_i(x) \tag{4.3}$$

We construct the followings

$$\mathbf{D}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 48 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}^{(5/2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 18.7230 & 13.8488 & 2.3404 & -0.5770 \end{pmatrix}$$

The proposed strategy in Section 4.1 gives us

$$C^T D^3 \Psi(x) + C^T D^{5/2} \Psi(x) + [C^T \Psi(x)]^2 - x^4 = 0 \tag{4.4}$$

and with the initial conditions we get

$$\begin{array}{rcl} c_0 - 2c_2 & = & 0 \\ 2c_1 - 12c_3 & = & 0 \\ 8c_2 & = & 2 \end{array} \tag{4.5}$$

After we solve the system of equations (4.4)-(4.5) we obtain $c_0 = 1/2$, $c_1 = 0$, $c_2 = 1/4$ and $c_3 = 0$ and the solution is

$$u(x) = C^T \Psi(x) = x^2$$

the same as exact solution.

Example 4.2. *Our second example is [9]*

$$D^4 u(x) + D^{7/2} u(x) + u^3(x) = x^9 \quad u(0) = 0, u'(0) = 0, u''(0) = 0, u'''(0) = 6 \tag{4.6}$$

The exact solution of the problem is $u(x) = x^3$.

We consider the solution for $N = 4$ and then

$$\mathbf{D}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 384 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{D}^{(7/2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 149.7839 & 110.7905 & 18.7230 & -4.6163 & -1.1702 \end{pmatrix}$$

If we apply Equation (3.18) with conditions we get

$$
\begin{aligned}
38c_4 + 98.29c_4.(c_0 - 2c_2 + 12c_4)^2 &= 0 \\
c_0 - 2c_2 + 12c_4 &= 0 \\
2c_1 - 12c_3 &= 0 \\
8c_2 - 96c_4 &= 0 \\
48c_3 &= 6
\end{aligned}
\tag{4.7}
$$

By solving this system (4.7) we obtain $c_0 = 0$, $c_1 = 0.75$, $c_2 = 0$, $c_3 = 0.125$ and $c_4 = 0$. Therefore our solution is

$$u(x) = C^T \Psi(x) = x^3.$$

which is the same as the exact solution.

Example 4.3. *Our next example is*

$$
D^\lambda u(x) + Du(x)D^\eta u(x) + u^2(x) = x^6 + \frac{6x^{(3-\lambda)}}{\Gamma(4-\lambda)} + \frac{36x^{(6-\nu-\eta)}}{\Gamma(4-\eta)\Gamma(4-\nu)}
$$

$$
u(0) = 0, u'(0) = 0, u''(0) = 6
\tag{4.8}
$$

The following conditions are given as: $\lambda \in (2,3), \nu \in (1,2), \eta \in (0,1)$

The solution will be found for $N = 3$ and for the collocation point one root of the $H_3(x)$ is used. Specially we take $\lambda = 2.5, \nu = 1.5, \eta = 0.9$.

$$28.0845c_3 + (1.0543c_1 + 4.4087c_2 + 10.6522c_3 + 6^{(1/2)} * (0.5773c_1 +$$

$$33.9225c_3 + (4.6807c_2 + 23.0813c_3 + 6^{(0.5)} * (2.3081c_2 + 9.3615c_3)) * (1.0543c_1 +$$
$$4.4087c_2 + 10.6522c_3 + 6^{(0.5)} * (0.5773c_1 + 1.9169c_2 + 2.8342c_3)) = 0
\tag{4.9}$$

And from the conditions we have these equations:

$$
\begin{aligned}
c_0 - 2c_2 &= 0 \\
2c_1 - 12c_3 &= 0 \\
48c_3 &= 6
\end{aligned}
\tag{4.10}
$$

After solving these systems we get $c_0 = 0, c_1 = 0.75, c_2 = 0, c_3 = 0.125$ Our approximate solution is the same as exact solution which is $u(x) = x^3$.

Example 4.4. *We then consider the following Riccati equation [10]*

$$
D^\alpha u(x) + u^2(x) - 2u(x) = 1, 0 < \alpha, x \le 1
\tag{4.11}
$$

$$
u(0) = 0,
\tag{4.12}
$$

The exact solution for this equations with $\alpha = 1$ is given as

$$
u(x) = 1 + 2tanh\left(2x + \frac{1}{2}log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)
\tag{4.13}
$$

By using our method, we solve this equation for $\alpha = 0.9$ and $\alpha = 1$. The results of the exact and two approximated solutions are plotted in Fig. 1. The results show that approximated solution tends to converge to the exact one when α is close to 1. When compared to results presented in [10], we achieve a good level of error.
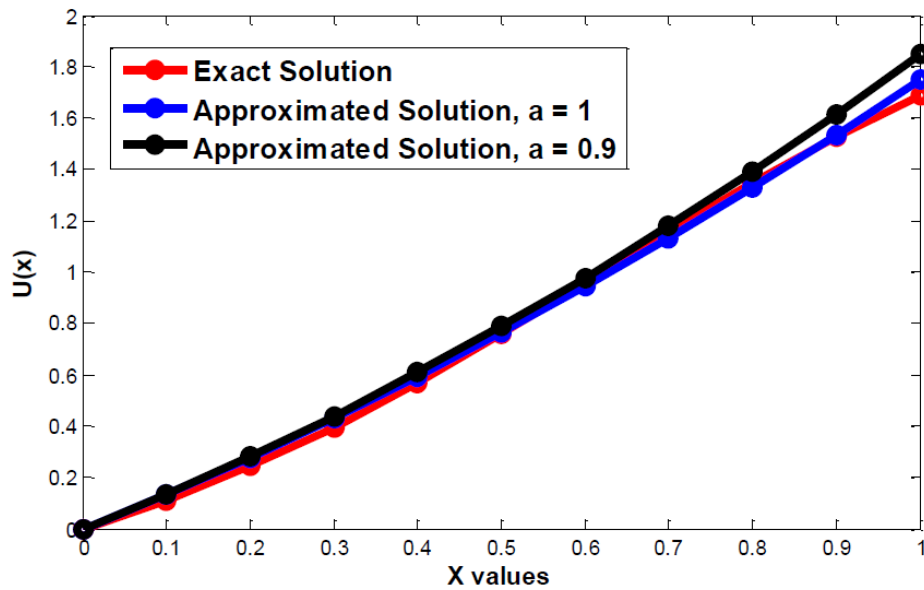
Fig. 1. The results of exact and approximated solutions

5 Conclusions

In this paper, we generated the operational matrix of fractional derivative by Hermite polynomials for more accurate and reliable solutions of nonlinear FDEs. With this matrix, the problem of nonlinear FDEs is redefined as a system of algebraic equations for simplicity. These algebraic equations including pre-given initial conditions are easily solved by the proposed approach. It was observed that the proposed approach is highly efficient in both numerical and analytical solutions of nonlinear types of FDEs through various examples.

Competing Interests

Authors have declared that no competing interests exist.

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