Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 3234263, 9 pages
https://doi.org/10.1155/2021/3234263

Research Article

Uniqueness of Complete Hypersurfaces in Weighted Riemannian Warped Products

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Received 6 July 2020; Revised 18 December 2020; Accepted 22 December 2020; Published 2 January 2021

Academic Editor: Leopoldo Greco

In this paper, applying the weak maximum principle, we obtain the uniqueness results for the hypersurfaces under suitable geometric restrictions on the weighted mean curvature immersed in a weighted Riemannian warped product $I \times_{\rho} M^n_f$ whose fiber M has f-parabolic universal covering. Furthermore, applications to the weighted hyperbolic space are given. In particular, we also study the special case when the ambient space is weighted product space and provide some results by Bochner's formula. As a consequence of this parametric study, we also establish Bernstein-type properties of the entire graphs in weighted Riemannian warped products.

1. Introduction

In recent years, the study of complete hypersurfaces in Riemannian manifolds has attracted many geometers. This is due to the fact that such hypersurfaces exhibit nice Bernstein-type properties.

Particularly, from the geometric analysis point of view, many problems lead us to consider Riemannian manifolds with a measure that has a positive smooth density with respect to the Riemannian one. This turns out to be compatible with the metric structure of the manifold, and the resulting spaces are the *weighted manifolds*, which are also called manifolds with density or smooth metric measure spaces. More precisely, the weighted manifold M^n_f is associated with a complete n-dimensional Riemannian manifold (M^n, g), and a smooth function f on M^n is the triple $(M^n, g, d\mu = e^{-f} dM)$, where dM stands for the volume element of M^n. In this setting, we will take into account the so-called Bakry-Émery Ricci tensor (see [1]) which is an extension of the standard Ricci tensor Ric, which is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f. \tag{1}$$

So, it is natural to extend some results of the Ricci curvature to analogous results for the Bakry-Émery Ricci tensor. Before giving more details on our work, we present a brief outline of some recent results related to ours.

In [2], Wei and Wylie studied the complete n-dimensional weighted Riemannian manifold and proved the weighted mean curvature and volume comparison results under the ∞-Bakry-Émery Ricci tensor is bounded from below and f or $|\nabla f|$ is bounded. Later, de Lima et al. [3, 4] researched the uniqueness of complete two-sided hypersurfaces immersed in weighted warped products by applying the appropriated generalized maximum principles. Moreover, [5] established Liouville-type results related to two-sided hypersurfaces immersed in a weighted Killing warped product. More recently, some uniqueness results of complete two-sided hypersurfaces in warped products with density are given in [6].

In this paper, we study complete hypersurfaces in a weighted Riemannian warped product. The Riemannian warped product $I \times_{\rho} M^n$ where $I \subset \mathbb{R}$ is an open interval, M^n is a complete n-dimensional Riemannian manifold, $\rho : I \longrightarrow \mathbb{R}^+$ is a positive smooth warping function, and the warped metric is given by

$$\langle , \rangle = dt^2 + \rho(t)^2 \langle , \rangle_M, \tag{2}$$

where $\langle , \rangle_M$ is the metric tensor of M^n. Furthermore, there exists a distinguished family of hypersurfaces in Riemannian warped products, that is so-called slices, which are defined as level hypersurfaces of the coordinate of the space. Notice that any slice is totally umbilical and has constant mean curvature.

This manuscript is organized as follows. In Section 2, we introduce some basic notions and facts of the hypersurfaces immersed in weighted Riemannian warped products. Section 3 is devoted to prove some results concerning the f-parabolicity of weighted manifolds and pay attention to show the weak maximum principle for the f-Laplace operator Δ_f holds on f-parabolic weighted manifolds. Moreover, by using the weak maximum principle, we provide the sign relationship among the f-mean curvature and the derivative of the warping function. These auxiliary results will be the key to obtaining our results. In our main results, we establish the uniqueness results for complete hypersurfaces under appropriate conditions on the f-mean curvature and the warping function in weighted Riemannian warped products $\bar{M}^{n+1} = I \times_\rho M_f^n$ whose fiber M_f^n has f-parabolic universal covering. Besides, we also present some applications related to our results. In Section 4, applying the weak maximum principle and Bochner's formula, we obtain some rigidity results for the special case when the ambient space is weighted product space. Section 5, as a nondirect application of our parametric case, we get nonparametric results for the entire graphs in weighted Riemannian warped products.

2. Preliminaries

Let M^n be a connected n-dimensional oriented Riemannian manifold and $I \subset \mathbb{R}$ be an open interval which is endowed with the metric dt^2. Let $\rho : I \longrightarrow \mathbb{R}^+$ be a positive smooth function. Denote $I \times_\rho M^n$ to be the product manifold with the following Riemannian metric

$$\langle , \rangle = \pi_I^* \left(dt^2 \right) + \rho(\pi_I)^2 \pi_M^* (\langle , \rangle_M), \tag{3}$$

where π_I and π_M are the projections onto I and M, respectively. Following the terminology used in [7], Chap.7, this resulting space is a *warped product* with *fiber* $(M, \langle , \rangle_M)$, *base* (I, dt^2), and *warping function* ρ. Furthermore, for a fixed point $t_0 \in I$, we say that $M_{t_0}^n = \{t_0\} \times M^n$ is a *slice* of $I \times_\rho M^n$.

Recalling that a smooth immersion $\psi : \Sigma^n \longrightarrow I \times_\rho M^n$ of an n-dimensional connected manifold Σ^n is called to be a *hypersurface*. Moreover, the induced metric via ψ on Σ^n will be also denoted for $\langle , \rangle$.

Throughout this paper, we assume that Σ^n is a *two-sided hypersurface*. Recalling that a hypersurface Σ^n is called a two-sided hypersurface if its normal bundle is trivial, which means that there exists a globally defined unit normal vector field $N \in \mathfrak{X}^\perp(\Sigma)$. For instance, every hypersurface with never vanishing mean curvature is trivially two-sided. Moreover, when the hypersurface Σ^n is two-sided, a choice of N on Σ^n makes the second fundamental form globally defined on $\mathfrak{X}^\perp(\Sigma)$. In the sequel, the Riemannian warped product is clearly orientable. This allows us to take, for each two-sided hypersurface Σ^n, a unique unitary normal vector field N globally defined on Σ^n in the same orientation of the vector field ∂_t, $\partial_t := \partial / \partial t$, i.e., such that $\langle N, \partial_t \rangle \le 0$. By the wrong-way Cauchy-Schwarz inequality (see [7], Proposition 5.30), we have $-1 \le \langle N, \partial_t \rangle \le 0$, and the first equality holds at a point $p \in \Sigma^n$ if and only if $N = -\partial_t$ at p. Moreover, we will refer to the function $\Theta : \Sigma \longrightarrow [-1, 0]$, $\Theta := \langle N, \partial_t \rangle$, as the *angle function*. On the other hand, we will represent a particular function naturally attached to the hypersurface Σ^n by the *height function* $h = (\pi_I)|_\Sigma : \Sigma^n \longrightarrow I$.

It can be easily seen that a hypersurface in Riemannian warped products is a slice if and only if the height function is constant. We also observe that slice $\{t_0\} \times M^n$ of $I \times_\rho M^n$ has constant mean curvature $H = \rho'(t_0)/\rho(t_0)$ with respect to the unit normal vector field $N = -\partial_t$.

Let $\bar{\nabla}$ and ∇ stand for gradients with respect to the metrics of $I \times_\rho M^n$ and Σ^n, respectively. In a simple computation, we have

$$\bar{\nabla}\pi_I = \langle \bar{\nabla}\pi_I, \partial_t \rangle \partial_t = \partial_t. \tag{4}$$

So, the gradient of h on Σ^n is

$$\nabla h = \left(\bar{\nabla}\pi_I \right)^T = \partial_t^T = \partial_t - \Theta N. \tag{5}$$

Particularly, we have

$$|\nabla h|^2 = 1 - \Theta^2, \tag{6}$$

where $||$ denotes the norm of a vector field on Σ^n.

Now, we consider that a Riemannian warped product $I \times_\rho M^n$ endowed with a weighted function f, which will be called a weighted Riemannian warped product $I \times_\rho M_f^n$. In this setting, for a two-sided hypersurface Σ^n immersed into $I \times_\rho M_f^n$, the *f -divergence operator* on Σ^n is defined by

$$\mathrm{div}_f(X) = e^f \ \mathrm{div} \left(e^{-f} X \right), \tag{7}$$

where X is a tangent vector field on Σ^n.

For a smooth function $u : \Sigma^n \longrightarrow \mathbb{R}$, we define its *drifting Laplacian* by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle, \tag{8}$$

we will also denote such an operator as the f-Laplacian of Σ^n.

According to Gromov [8], the weighted mean curvature, or f-mean curvature H_f of Σ^n, is given by

$$nH_f = nH + \langle \bar{\nabla} f, N \rangle, \tag{9}$$

where H is the standard mean curvature of hypersurface Σ^n with respect to the Gauss map N.

Notice it follows from a splitting theorem by the case (see [9], Theorem 1.2) that if a weighted Riemannian warped product $I \times_\rho M_f^n$ is endowed with a bounded weighted function f and such that $\overline{\mathrm{Ric}}_f(V,V)\geq 0$ for all vector fields V on $I\times_\rho M_f^n$, then f must be constant along $\mathbb{R}$. So, motivated by this result, in the following, we will consider weighted Riemannian warped products $I\times_\rho M_f^n$ whose weighted function f does not depend on the parameter $t\in I$, that is $\langle\bar{\nabla}f,\partial_t\rangle=0$. Moreover, for simplicity, we will refer to them as $\bar{M}^{n+1}\coloneqq I\times_\rho M_f^n$.

Remark 1. We note that the f-mean curvature H_f of a slice $\{t_0\}\times M^n$ in a weighted Riemannian warped product $\bar{M}^{n+1}$ is given by,

$$H_f=\frac{\rho'(t_0)}{\rho(t_0)}. \tag{10}$$

Indeed, since $-\partial_t=N$ is a normal vector field to the slice $\{t_0\}\times M^n$, from (9), we have that $H_f=H=\rho'(t_0)/\rho(t_0)$.

For the proof of our main results in this paper, we need the following formulas that will be the extensions of Lemma 2 in [10].

Lemma 2 ([10]). *Let $\psi:\Sigma^n\longrightarrow I\times_\rho M_f^n$ be a hypersurface immersed in a weighted Riemannian warped product, with height function h. Then*

$$\Delta_f\sigma(h)=n\rho(h)\left(\frac{\rho'(h)}{\rho(h)}+H_f\Theta\right), \tag{11}$$

$$\Delta_f h=(\log\rho)'(h)\left(n-|\nabla h|^2\right)+nH_f\Theta, \tag{12}$$

$$\Delta_f\rho(h)=n\frac{\rho'(h)^2}{\rho(h)}+\rho(h)(\log\rho)''(h)|\nabla h|^2+n\rho'(h)H_f\Theta, \tag{13}$$

where $\sigma(t)$ is a primitive function of $\rho(t)$.

In the following terminology introduced in [11], we present the definition of the weak maximum principle for the drifted Laplacian. The next lemma extended the result of [11].

Lemma 3 ([11]). *Let $(M^n,\langle,\rangle_M,e^{-f}dM)$ be an n-dimensional (not necessarily complete) weighted Riemannian manifold. We say that the weak maximum principle for the f-Laplace operator Δ_f holds on M, if for any smooth bounded above function u on M, there exists a sequence $\{p_j\}\subset M$ such that*

$$\lim_j u\left(p_j\right)=\sup u, \text{and} \lim_j\Delta_f u\left(p_j\right)\leq 0. \tag{14}$$

Equivalently, for any smooth bounded below function u on M, then there is a sequence $\{q_j\}\subset M$ such that

$$\lim_j u\left(q_j\right)=\inf u, \lim_j\Delta_f u\left(q_j\right)\geq 0. \tag{15}$$

On the other hand, a smooth function u on a weighted manifold M_f is called f-superharmonic if $\Delta_f u\leq 0$. Taking this into account, a noncompact weighted manifold $(M^n,,,e^{-f}dM)$ is said to be f-parabolic if it does not admit nonconstant positive f-superharmonic functions on M. So, we can conclude the following extension of Theorem 1 in [12], which establishes sufficient conditions to ensure that the two-sided hypersurface Σ^n in $\bar{M}^{n+1}$ is f-parabolic.

Lemma 4 ([12]). *Let Σ^n be a complete two-sided hypersurface in a weighted Riemannian warped product $I\times_\rho M_f^n$ whose fiber M has f-parabolic universal covering. If the angle function Θ is bounded and the restriction $\rho(h)$ on Σ^n of warping function ρ satisfies:*

(c_1) $\sup\rho(h)<\infty$

(c_2) $\inf\rho(h)>0$

then, Σ^n is f-parabolic.

3. Uniqueness Results in Weighted Riemannian Warped Products

In this section, we will study the uniqueness for complete hypersurfaces in weighted Riemannian warped products $\bar{M}^{n+1}$. Before describing our main results, we will prove some auxiliary propositions which will be essential in the sequel.

Proposition 5. *If the weighted manifold $(M^n,\langle,\rangle_M,e^{-f}dM)$ is f-parabolic, then the weak maximum principle for the f -Laplace operator Δ_f holds on M.*

Proof. Since weighted manifold M is f-parabolic, using Corollary 6.4 in [13] it follows that M is also stochastically complete.

On the other hand, by the fact which in [11] that M satisfies the weak maximum principle for the f-Laplace operator Δ_f if and only if M is stochastically complete, this concludes the proof.

Furthermore, for any compact subset $\Omega\subset(M^n,\langle,\rangle_M,e^{-f}dM)$, we define the f-capacity of Ω as,

$$\mathrm{cap}_f(\Omega)=\inf\left\{\int_M|\nabla u|^2e^{-f}dM : u\in\mathrm{Lip}_f(M), u\Big|_\Omega\equiv 1\right\}, \tag{16}$$

where $\mathrm{Lip}_f(M)$ is the set of all compactly supported Lipschitz functions on M. By the fact that a weighted manifold is f-parabolic if and only if $\mathrm{cap}_f(\Omega)=0$ for any compact set Ω.

The following lemma is the extension of Lemma 3 in [14], which will allow us to obtain our technical result.

Lemma 6 ([14]). *Let $(M^n,\langle,\rangle_M,e^{-f}dM)$ be an $n(\geq 2)$ -dimensional weighted manifold and consider $\nu\in C^2(M)$ which* *satisfies $v\Delta_f v \geq 0$. Let B_R be a geodesic ball of radius R around $p \in M$. For any r such that $0 < r < R$, we have*

$$\int_{B_r} |\nabla v|^2 e^{-f} dM \leq \frac{4 \sup_{B_R} v^2}{\mu_{r,R}}, \tag{17}$$

where B_r denotes the geodesic ball of radius r around $p \in M$ and $1/\mu_{r,R}$ is the f-capacity of the annulus $B_R \setminus \bar{B}_r$.

Proposition 7. *Let $(M^n, \langle,\rangle_M, e^{-f} dM)$ be an f-parabolic weighted manifold and $u \in C^2(M)$ be a positive function on M and $u^* = \sup_M u < +\infty$. If $\Delta_f u$ does not change the sign on M, then u is constant on M.*

Proof. Since $\Delta_f u$ does not change the sign on M, thus $\Delta_f u \geq 0$ or $\Delta_f u \leq 0$. If $\Delta_f u \geq 0$, then $u\Delta_f u \geq 0$. Considering u is bounded from above and $u > 0$, we shall find a positive constant C such that $u^2 \leq C$ on M. For a geodesic ball B_R of radius R around $p \in M$, by Lemma 6, for any r such that $0 < r < R$, we have that the function u satisfies

$$\int_{B_r} |\nabla u|^2 e^{-f} dM \leq \frac{4C}{\mu_{r,R}}. \tag{18}$$

Taking into account that M is f-parabolic, we know that $1/\mu_{r,R} \longrightarrow 0$ as $R \longrightarrow \infty$, that is, $|\nabla u|^2$ vanishes identically on M. So, u is constant on M.

On the other hand, when $\Delta_f u \leq 0$, it follows that u is a f-superharmonic function on M, which is bounded from above. So, the conclusion now follows from f-parabolicity.

In the following, applying the weak maximum principle, we provide the sign relationship between the f-mean curvature and the derivative of warping function, in which the results extend the Lemma 14 in [15]. We point out that, to prove the following results, we do not require that the f-mean curvature H_f of the hypersurface Σ^n is constant.

First, recall that a slab of a weighted Riemannian warped product $I \times_\rho M^n_f$ is a region of the type

$$[t_1, t_2] \times M^n_f = \left\{(t, p) \in I \times_\rho M^n_f : t_1 \leq t \leq t_2\right\}. \tag{19}$$

Proposition 8. *Let $\psi : \Sigma^n \longrightarrow I \times_\rho M^n_f$ be a hypersurface with nonvanishing f-mean curvature which is contained in a slab. Choose on Σ the orientation such that $H_f > 0$. Assume that the weak maximum principle for the f -Laplace operator Δ_f holds on Σ. If either*

(i) The warping function ρ is monotonic or

(ii) The function $(\log \rho)'$ is nondecreasing

then $\rho'(h) \geq 0$. On the other hand, if $H_f < 0$, then $\rho'(h) \leq 0$.

Proof. Since the hypersurface Σ^n is contained in a slab, then the height function h is bounded and $\sup_\Sigma \sigma(h) = \sigma(h^*)$, $\inf_\Sigma \sigma(h) = \sigma(h_*)$, where $h^* = \sup_\Sigma h$, $h_* = \inf_\Sigma h$. Applying the weak maximum principle to the f-Laplacian $\Delta_f \sigma(h)$, we may find two sequences $\{p_j\}, \{q_j\} \subset \Sigma^n$ such that

$$\begin{aligned} &\lim_j \sigma\left(h\left(p_j\right)\right) = \sigma(h^*), \lim_j \Delta_f \sigma\left(h\left(p_j\right)\right) \leq 0, \\ &\lim_j \sigma\left(h\left(q_j\right)\right) = \sigma(h_*), \lim_j \Delta_f \sigma\left(h\left(q_j\right)\right) \geq 0. \end{aligned} \tag{20}$$

From (11), we have $\Delta_f \sigma(h) = n\rho(h)((\rho'/\rho)(h) + H_f \Theta)$, then

$$\lim_j \Delta_f \sigma\left(h\left(p_j\right)\right) = \lim_j n\rho\left(h\left(p_j\right)\right)\left(\frac{\rho'}{\rho}\left(h\left(p_j\right)\right) + \Theta\left(p_j\right) H_f\left(p_j\right)\right) \leq 0, \tag{21}$$

$$\lim_j \Delta_f \sigma\left(h\left(q_j\right)\right) = \lim_j n\rho\left(h\left(q_j\right)\right)\left(\frac{\rho'}{\rho}\left(h\left(q_j\right)\right) + \Theta\left(q_j\right) H_f\left(q_j\right)\right) \geq 0. \tag{22}$$

(i) Since $H_f > 0$, from (22), we get

$$\lim_j \frac{\rho'}{\rho}\left(h\left(q_j\right)\right) \geq \lim_j -\Theta\left(q_j\right) H_f\left(q_j\right) \geq 0, \tag{23}$$

where the last inequality is due to $-1 \leq \Theta \leq 0$. Furthermore, taking into account that $\rho(h)$ is monotonic, therefore $\rho'(h) \geq 0$.

On the other hand, since $H_f < 0$ and $-1 \leq \Theta \leq 0$, jointly with (21), we have

$$\lim_j \frac{\rho'}{\rho}\left(h\left(p_j\right)\right) \leq \lim_j -\Theta\left(p_j\right) H_f\left(p_j\right) \leq 0. \tag{24}$$

So $\rho'(h) \leq 0$ follows from that $\rho(h)$ is monotonic.

(ii) Since $H_f > 0$, $-1 \leq \Theta \leq 0$, and $(\log \rho)'$ is nondecreasing, it follows from (22) that

$$\frac{\rho'}{\rho}(h_*) \geq \lim_j -\Theta\left(q_j\right) H_f\left(q_j\right) \geq 0. \tag{25}$$

Therefore,

$$\frac{\rho'}{\rho}(h) \geq \frac{\rho'}{\rho}(h_*) \geq 0. \tag{26}$$

So $\rho'(h) \geq 0$.

Now assume that $H_f < 0$, from (21), we have

$$\frac{\rho'}{\rho}(h^*) \le \lim_j -\Theta\left(p_j\right)H_f\left(p_j\right) \le 0. \tag{27}$$

Therefore, we conclude that

$$\frac{\rho'}{\rho}(h) \le \frac{\rho'}{\rho}(h^*) \le 0. \tag{28}$$

So $\rho'(h) \le 0$.

After the following theorem, we derive our uniqueness results for parabolic hypersurfaces.

Theorem 9. *Let $\bar{M}^{n+1} = I \times_\rho M_f^n$ be a weighted Riemannian warped product whose fiber M has f-parabolic universal covering. Let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ be a complete hypersurface with nonvanishing f-mean curvature which lies in a slab. Suppose the warping function $\rho(h)$ satisfies conditions (i) or (ii). If $H_f^2 \le (\rho'^2/\rho^2)(h)$, then Σ^n is a slice.*

Proof. Since M has f-parabolic universal covering, Σ lies in a slab and $-1 \le \Theta \le 0$; then, we deduce that Σ is f-parabolic by Lemma 4. Moreover, from Proposition 5, it follows that weak maximum principle for the f-Laplace operator Δ_f holds on Σ. Proceeding as above and considering the assumption that the warping function $\rho(h)$ satisfies conditions (i) or (ii), we have that Proposition 8 holds true.

In the case where $H_f > 0$, by Proposition 8, we have $\rho'(h) \ge 0$. Combining the assumption $H_f^2 \le (\rho'^2/\rho^2)(h)$, we obtain $H_f \le (\rho'/\rho)(h)$. Therefore, from (11), we have

$$\Delta_f\sigma(h) = n\rho(h)\left(\frac{\rho'}{\rho}(h) + H_f\Theta\right) \ge n\rho(h)\left(\frac{\rho'}{\rho}(h) - H_f\right) \ge 0, \tag{29}$$

where the first inequality is due to $\Theta \ge -1$.

Moreover, since $\sigma(h)$ is a positive smooth function and there is a constant C such that $\sigma(h) \le C$. From Proposition 7, we conclude that $\sigma(h)$, and hence, h is constant. Consequently, Σ^n is a slice.

Finally, in the case where $H_f < 0$, we know from Proposition 8 that $\rho'(h) \le 0$, so that $H_f \ge (\rho'/\rho)(h)$. Therefore,

$$\Delta_f\sigma(h) = n\rho(h)\left(\frac{\rho'}{\rho}(h) + H_f\Theta\right) \le n\rho(h)\left(\frac{\rho'}{\rho}(h) - H_f\right) \le 0. \tag{30}$$

The proof then follows as in the case $H_f > 0$.

Moreover, if warping function $\rho(h)$ satisfies condition (ii), then using (13), we have the next result which extends Theorem 9.

Theorem 10. *Let $\bar{M}^{n+1} = I \times_\rho M_f^n$ be a weighted Riemannian warped product whose fiber M has f-parabolic universal covering. Let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ be a complete two-sided hypersurface that lies in a slab. Suppose the warping function $\rho(h)$ satisfies conditions (ii). If the f-mean curvature H_f satisfies $H_f^2 \le (1/\Theta^2)(\rho'(h)^2/\rho(h)^2)$, then Σ^n is a slice.*

Proof. From (13), we have

$$\begin{aligned}\frac{1}{\rho(h)}\Delta_f\rho(h) &= n\frac{\rho'(h)^2}{\rho(h)^2} + (\log\rho)''(h)|\nabla h|^2 + n\frac{\rho'(h)}{\rho(h)}H_f\Theta \\ &\ge n\frac{\rho'(h)^2}{\rho(h)^2} + (\log\rho)''(h)|\nabla h|^2 - \frac{n}{2}\left(\frac{\rho'(h)^2}{\rho(h)^2} + H_f^2\Theta^2\right) \\ &\ge \frac{n}{2}\left(\frac{\rho'(h)^2}{\rho(h)^2} - H_f^2\Theta^2\right) + (\log\rho)''(h)|\nabla h|^2.\end{aligned} \tag{31}$$

By the hypothesis, we have $\Delta_f\rho(h) \ge 0$. Moreover, since Σ^n lies in a slab, and $\rho(h)$ is a positive smooth function, then there exists a positive constant C such that $\rho(h) \le C$. So, we can apply Proposition 7 to get $\rho(h)$ as constant. Therefore, Σ^n is a slice.

Now, we consider the $(n+1)$-dimensional weighted hyperbolic space $\mathbb{H}^{n+1}$, which instead of the more commonly used weighted half-space model, as the weighted warped product $\mathbb{R} \times_{e^t} \mathbb{R}_f^n$. It can be easily seen that the slices $\{t_0\} \times \mathbb{R}_f^n$ of $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}_f^n$ are precisely the horospheres. Furthermore, according to Theorem 10, we have the following application in weighted hyperbolic space.

Corollary 11. *Let $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}_f^n$ be a weighted hyperbolic space whose fiber $\mathbb{R}^n$ has f-parabolic universal covering and let $\psi : \Sigma^n \longrightarrow \mathbb{H}^{n+1}$ be a complete two-sided hypersurface which is contained in a slab. If $H_f^2 \le (1/\Theta^2)$, then Σ^n is a slice.*

Next, we will use the weak maximum principle to study another rigidity of the hypersurfaces in weighted Riemannian warped products.

Theorem 12. *Let $\bar{M}^{n+1} = I \times_\rho M_f^n$ be a weighted Riemannian warped product whose fiber M has f-parabolic universal covering. Let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ be a complete two-sided hypersurface which lies in a slab. Suppose the warping function $\rho(h)$ satisfies condition (ii), and there is a point $h_0 \in I$ such that $\rho'(h_0) = 0$. If H_f does not change sign, then $H_f = 0$ and Σ^n is a slice.*

Proof. Since the hypersurface Σ^n is contained in a slab, then h is bounded, and $\sup_\Sigma h = h^*$, $\inf_\Sigma h = h_*$. Reasoning as in the proof of Theorem 9, we have the weak maximum principle for f-Laplace operator Δ_f holds on Σ^n; then, there exist two sequences $\{p_j\}, \{q_j\} \subset \Sigma^n$ such that

$$\begin{aligned}\lim_j h(p_j) &= h^*, \lim_j \Delta_f h(p_j) \le 0,\\ \lim_j h(q_j) &= h_*, \lim_j \Delta_f h(q_j) \ge 0.\end{aligned} \quad (32)$$

From (12), we have that

$$\lim_j \Delta_f h(p_j) = \lim_j \frac{\rho'}{\rho}(h(p_j))\left(n - \left|\nabla h(p_j)\right|^2\right) + n\Theta(p_j)H_f(p_j) \le 0, \quad (33)$$

$$\lim_j \Delta_f h(q_j) = \lim_j \frac{\rho'}{\rho}(h(q_j))\left(n - \left|\nabla h(q_j)\right|^2\right) + n\Theta(q_j)H_f(q_j) \ge 0. \quad (34)$$

Take $-1 \le \Theta < 0$. Thus, it follows from (33) and (34) that

$$H_f(p_j) \ge \frac{\left(-\rho'(h^*)/\rho(h^*)\right)\left\{n - \left|\nabla h(p_j)\right|^2\right\}}{n\Theta(p_j)}, \quad (35)$$

and

$$H_f(q_j) \le \frac{\left(-\rho'(h_*)/\rho(h_*)\right)\left\{n - \left|\nabla h(q_j)\right|^2\right\}}{n\Theta(q_j)}. \quad (36)$$

Furthermore, taking into account that (ρ'/ρ) is nondecreasing yields

$$\frac{\rho'}{\rho}(h_*) \le \frac{\rho'}{\rho}(h_0) \le \frac{\rho'}{\rho}(h^*). \quad (37)$$

On the other hand, by (6), we have $n - |\nabla h|^2 > 0$. Therefore, using (35) and (36), we conclude that

$$\begin{aligned} H_f(p_j) &\ge \frac{\left(-\rho'(h_0)/\rho(h_0)\right)\left\{n - \left|\nabla h(p_j)\right|^2\right\}}{n\Theta(p_j)} = 0,\\ H_f(q_j) &\le \frac{\left(-\rho'(h_0)/\rho(h_0)\right)\left\{n - \left|\nabla h(q_j)\right|^2\right\}}{n\Theta(q_j)} = 0. \end{aligned} \quad (38)$$

Considering that H_f does not change the sign on Σ^n, hence $H_f = 0$, that is, Σ^n is a f-minimal hypersurface. Using (13), we have

$$\Delta_f \rho(h) = n\frac{\rho'(h)^2}{\rho(h)} + |\nabla h|^2 \rho(h)(\log \rho)''(h) \ge 0. \quad (39)$$

In the following, by the same argument as in Theorem 10, we have Σ^n is a slice.

4. Uniqueness Results in Weighted Product Spaces

In this section, we establish some uniqueness results concerning the complete hypersurfaces Σ^n in weighted product spaces $I \times M^n_f$. Firstly, as a consequence of Theorem 9, when the ambient space is weighted product space, we have the following result.

Corollary 13. *Let $\bar{M}^{n+1} = I \times M^n_f$ be a weighted product space whose fiber M has f -parabolic universal covering, and let $\psi : \Sigma^n \longrightarrow \bar{M}^{n+1}$ be a complete hypersurface with nonvanishing f -mean curvature which is contained in a slab. If H_f does not change the sign on Σ^n, then Σ^n is a slice.*

Proof. From (12), we have

$$\Delta_f h = nH_f\Theta. \quad (40)$$

Since H_f does not change sign, we have $\Delta_f h$ does not change the sign on Σ^n, and proceeding as in the proof of Theorem 9, we obtain that Σ^n is parabolic. Furthermore, we know that height function h is bounded, which implies that Σ^n is a slice from Proposition 7.

Theorem 14. *Let $\psi : \Sigma^n \longrightarrow I \times M^n_f$ be a complete hypersurface with nonvanishing constant f -mean curvature in a weighted product space $I \times M^n_f$ whose fiber M has f -parabolic universal covering. Assume that $K_M \ge k$ for some nonnegative constant k and the weighted function f is convex. If $-1 \le \Theta \le -(\sqrt{2}/2)$, then Σ^n is a slice.*

Proof. Let $E_1, \cdots, E_n$ be a (local) orthonormal frame in $\mathfrak{X}(\Sigma)$; using the Gauss equation, we have

$$\mathrm{Ric}(X, X) = \sum_{i=1}^{n} \left\langle \bar{R}(X, E_i)X, E_i \right\rangle + nH\langle AX, X\rangle - |AX|^2, \quad (41)$$

for any $X \in \mathfrak{X}(\Sigma)$. Moreover, we also have

$$\left\langle \bar{R}(X, E_i)X, E_i \right\rangle = K_M(X^*, E_i^*)\left(\langle X^*, X^*\rangle\langle E_i^*, E_i^*\rangle - \langle X^*, E_i^*\rangle^2\right), \quad (42)$$

where K_M is the sectional curvature of M, $X^* = X - \langle X, \partial_t\rangle\partial_t$ and $E_i^* = E_i - \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto M, respectively.

Considering the hypothesis that $K_M \geq k$ for some nonnegative constant k, and by direct computation, we have

$$\sum_{i=1}^{n} \langle \bar{R}(X, E_i)X, E_i \rangle \geq k\left((n-1)|X|^2 - (n-2)\langle X, \nabla h\rangle^2 - |X|^2|\nabla h|^2\right). \tag{43}$$

Summing up,

$$\begin{aligned}\operatorname{Ric}(X, X) \geq k\left((n-1)|X|^2 - (n-2)\langle X, \nabla h\rangle^2 - |X|^2|\nabla h|^2\right) \\ + nH\langle AX, X\rangle - |AX|^2.\end{aligned} \tag{44}$$

Furthermore, taking into account that the weighted function f is convex, we have

$$\operatorname{Hess} f(X, X) = \bar{\operatorname{Hess}} f(X, X) + \langle \bar{\nabla} f, N\rangle\langle AX, X\rangle \geq \langle \bar{\nabla} f, N\rangle\langle AX, X\rangle. \tag{45}$$

So

$$\begin{aligned}\operatorname{Ric}_f(X, X) \geq k\left((n-1)|X|^2 - (n-2)\langle X, \nabla h\rangle^2 - |X|^2|\nabla h|^2\right) \\ + nH_f\langle AX, X\rangle - |AX|^2.\end{aligned} \tag{46}$$

Particularly,

$$\operatorname{Ric}_f(\nabla h, \nabla h) \geq k(n-1)\left(1 - |\nabla h|^2\right)|\nabla h|^2 + nH_f\langle A\nabla h, \nabla h\rangle - |A\nabla h|^2. \tag{47}$$

On the other hand, using the Bochner-Lichnerowiz formula (see [2]),

$$\frac{1}{2}\Delta_f\left(|\nabla h|^2\right) = |\operatorname{Hess} h|^2 + \operatorname{Ric}_f(\nabla h, \nabla h) + \langle \nabla \Delta_f h, \nabla h\rangle, \tag{48}$$

where

$$\operatorname{Hess} h(X) = \nabla X \nabla h = \Theta AX, \text{ for any } X \in \mathfrak{X}(\Sigma). \tag{49}$$

Thus, from (5), we have that

$$|\operatorname{Hess} h|^2 = |A|^2\Theta^2 = |A|^2\left(1 - |\nabla h|^2\right). \tag{50}$$

Using H_f as a constant and (12), we have

$$\nabla \Delta h = nH_f \nabla(\Theta) = -nH_f A(\nabla h). \tag{51}$$

Consequently,

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq k(n-1)\left(1 - |\nabla h|^2\right)|\nabla h|^2\big) + |A|^2\left(1 - 2|\nabla h|^2\right). \tag{52}$$

Next, by the hypothesis $-1 \leq \Theta \leq -(\sqrt{2}/2)$ and (6), we have $|\nabla h|^2 \leq (1/2)$. So, from (52), it follows that

$$\Delta_f|\nabla h|^2 \geq 2k(n-1)\left(1 - |\nabla h|^2\right)|\nabla h|^2 \geq 0. \tag{53}$$

Moreover, reasoning as in the proof of Theorem 9, we have that Σ^n is parabolic. Since $|\nabla h|^2$ is bounded on Σ^n, thus by Proposition 7, we conclude that $|\nabla h|^2$ is constant. So, Δ_f $|\nabla h|^2 = 0$. From (53), we also have $|\nabla h|^2 = 0$ on Σ^n; consequently, h is constant, that is, Σ^n is a slice.

To prove our next result, we need the following auxiliary lemma.

Lemma 15 ([3]). *Let $\psi : \Sigma^n \longrightarrow I \times M^n_f$ be a hypersurface with constant f-mean curvature immersed in a weighted product space $I \times M^n_f$; then*

$$\Delta_f\Theta = -\left(|A|^2 + \widetilde{\operatorname{Ric}}_f(N^*, N^*)\right)\Theta, \tag{54}$$

where $\widetilde{\operatorname{Ric}}_f$ is the Bakry-Émery-Ricci tensor of the fiber M and $N^* = N - \langle N, \partial_t\rangle\partial_t$ is the orthonormal projection of N onto M.

Theorem 16. *Let $\psi : \Sigma^n \longrightarrow I \times M^n_f$ be a complete hypersurface with nonvanishing constant f-mean curvature in a weighted product space $I \times M^n_f$ whose fiber M has f-parabolic universal covering. Assume that $K_M \geq -k$ for some positive constant k and the weighted function f is convex. If $|\nabla h|^2 \leq ((\alpha|A|^2)/((n-1)k))$, for some constant $0 < \alpha < 1$, then Σ^n is a slice.*

Proof. By the Gauss equation and with a direct computation, we have

$$\widetilde{\operatorname{Ric}}_f(N^*, N^*) \geq -k(n-1)|\nabla h|^2. \tag{55}$$

In the case where $-1 \leq \Theta < 0$, from Lemma 15 and the condition $|\nabla h|^2 \leq ((\alpha|A|^2)/((n-1)k))$, we have

$$\Delta_f\Theta \geq -\left(|A|^2 - k(n-1)|\nabla h|^2\right)\Theta \geq -\left(\frac{1}{\alpha} - 1\right)k(n-1)|\nabla h|^2\Theta \geq 0. \tag{56}$$

On the other hand, as we did before in the proof of Theorem 9, it follows that Σ^n is parabolic. Moreover, since Θ is bounded on Σ^n, we conclude from Proposition 7 that Θ is constant. So, $\Delta_f\Theta = 0$. Using (56), we have $|\nabla h|^2 = 0$, which implies that h is constant. Therefore, Σ^n is a slice.

As a consequence of the proof of Theorem 16, we can get the following corollary.

Corollary 17. *Let $\psi : \Sigma^n \longrightarrow I \times M^n_f$ be a complete hypersurface with nonvanishing constant f -mean curvature in a weighted product space $I \times M^n_f$ whose fiber M has f -parabolic* *universal covering. Assume that $K_M \geq -k$ and $\bar{\text{Hess}}f \geq -\beta$ for some positive constants k and β. If $|\nabla h|^2 \leq ((\alpha|A|^2)/((n-1)k+\beta))$, for some constant $0<\alpha<1$, then Σ^n is a slice.*

5. Nonparametric Results for the Entire Graphs

In this section, we consider the vertical graphs in a weighted Riemannian warped product $\bar{M}^{n+1} = I \times_\rho M_f^n$, which are defined by

$$\Sigma^n(u) = \{(u(x), x) \colon x \in \Omega\} \subset \bar{M}^{n+1}, \tag{57}$$

where $\Omega \subseteq M$ be a connected domain of M and u is a smooth function on Ω. Moreover, the metric induced on Ω from the metric on ambient space $\bar{M}^{n+1}$ via $\Sigma^n(u)$, which is represented by

$$\langle , \rangle = du^2 + \rho^2(u)\langle , \rangle_M. \tag{58}$$

It is easy to see from the metric induced on Ω of $\Sigma^n(u)$ that if the function $\rho(u)$ is bounded on Ω, the graphs $\Sigma^n(u)$ is complete. Furthermore, the graph $\Sigma^n(u)$ is said to be entire if $\Omega = M$.

In the following, we can give the reason as in the proof of the Theorem 9 to obtain a nonparametric result.

Corollary 18. *Let $\bar{M}^{n+1} = I \times_\rho M_f^n$ be a weighted Riemannian warped product whose fiber M has f-parabolic universal covering, and let $\Sigma^n(u)$ be an entire graph with nonvanishing f-mean curvature which lies in a slab. Suppose the warping function $\rho(h)$ satisfies condition (i) or (ii). If $H_f^2 \leq ({\rho'}^2/\rho^2)(h)$, then $\Sigma^n(u)$ is a slice.*

Next, it is not difficult to obtain the following nonparametric version of Theorem 10.

Corollary 19. *Let $\bar{M}^{n+1} = I \times_\rho M_f^n$ be a weighted Riemannian warped product whose fiber M has f-parabolic universal covering. Let $\Sigma^n(u)$ be an entire graph which lies in a slab. Suppose the warping function $\rho(h)$ satisfies condition (ii). If the f-mean curvature H_f satisfies $H_f^2 \leq (1/\Theta^2)((\rho'(h)^2)/(\rho(h)^2))$, then $\Sigma^n(u)$ is a slice.*

If, moreover, $\rho = 1$, we can have the following corollaries of all other theorems of Section 4.

Corollary 20. *Let $\Sigma^n(u)$ be an entire graph with nonvanishing constant f-mean curvature in a weighted product space $I \times M_f^n$ whose fiber M has f-parabolic universal covering. Assume that $K_M \geq k$ for some nonnegative constant k and the weighted function f is convex. If $-1 \leq \Theta \leq -(\sqrt{2}/2)$, then $\Sigma^n(u)$ is a slice.*

Corollary 21. *Let $\Sigma^n(u)$ be an entire graph with nonvanishing constant f-mean curvature in a weighted product space $I \times M_f^n$ whose fiber M has f-parabolic universal covering. Assume that $K_M \geq -k$ for some positive constant k and the weighted function f is convex. If $|\nabla h|^2 \leq ((\alpha|A|^2)/((n-1)k))$, for some constant $0<\alpha<1$, then $\Sigma^n(u)$ is a slice.*

Data Availability

No data used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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