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# On Solvable Potentials, Supersymmetry, and the One-Dimensional Hydrogen Atom

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**Abstract:** The ways for improving on techniques for finding new solvable potentials based on supersymmetry and shape invariance has been discussed by Morales *et al.* [1] In doing so they address the peculiar system known as the one-dimensional hydrogen atom. In this paper we show that their remarks on such problem are mistaken. We do this by explicitly constructing both the one-dimensional Coulomb potential and the superpotential associated with the problem, objects whose existence are denied in the mentioned paper.

**Keywords:** one-dimensional hydrogen atom, one-dimensional Coulomb potential, supersymmetric quantum mechanics.

A paper of Morales *et. al.* [1] has discussed the use of supersymmetric and shape invariance techniques and of Darboux and intertwining transformations, for building new solvable potentials.

To illustrate these ideas they apply them to hydrogen-like potentials and to radial and one-dimensional problems. They assert [page 23 of [1], in the paragraph after Equation (39)] that the potential corresponding to a one-dimensional hydrogen atom, *i. e.* a one-dimensional Coulomb potential, is nonexistent. They further claim that there is no superpotential associated with the -1/|x| potential energy term [page 22 of [1], in the paragraph just before their Equation (36)]. In this letter we want to challenge these two affirmations. Throughout this work we use atomic units $q_e = \sim = m = 1$. In this paper we want to discuss their results concerning such 1D problem.

We recognize from the start that the potential deserving the name one-dimensional Coulomb potential is not the one usually alluded to in the literature—*i. e.* it is not -1/|x|. The true Coulomb potential in one dimension must be the solution of the corresponding Poisson equation

$$\nabla^2 \phi 1DC = -4\pi\delta(x) \qquad (1)$$

where δ(x) is a Dirac delta function which is really not a function but a distribution also termed a generalized function [2]. As it is very easy to realize, just solving Equation (3), the 1D Coulomb potential definitively exist and is given by

$$\phi 1DC = -2\pi|x| \qquad (2)$$

so the potential energy function needed in the Schrödinger equation should be

$$V_{1DC}(x) = 2\pi|x| \qquad (3)$$

In this sense the one-dimensional Coulomb potential does indeed exist. However, $V_{1DC}(x)$ is not the potential energy usually referred to as the one-dimensional hydrogen atom potential. But even if Morales *et. al.* are referring to this potential, namely $V_{1DH}$ = -1/|x|, corresponding to a Hamiltonian

$$H_{1DC} = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{|x|} \qquad (4)$$

the existence of a superpotential is beyond doubt, as we intend to exhibit in this work, see also [3,4]. The result [Equation (36) in [1]] they base their argument on the nonexistence of a superpotential for the one-dimensional hydrogen atom cannot be right since it does not have any explicit *r*-dependence. Even though this problem is surely just a misprint, the limit $l \to 0$ has no meaning for discrediting Hamiltonian (4) because the problem really comes from the need to describe Coulomb systems constrained to one-dimensional motions with no spherical symmetry and hence described by states with no well defined angular momentum.

Any system described by Hamiltonian (4) is one with baffling properties [5–7]. Its properties are so peculiar that people is prone to express erroneous concepts about it. For example, it has been claimed that the potntial energy term in Hamiltonian (4) is its own supersymmetric partner [8], or, as in [1], that the Hamiltonian itself cannot really be written since its potential energy function does not exist. On the other hand, it has been proven that it violateswhich the nondegeneracy theorem for one-dimensional quantum problems [5], and it has been shown that a superselection rule, analogous to the one preventing the so-called paradox of optical isomers of quantum chemistry, operates in the system [6,9–13]; see also [14,15] for other similar points of view. The Hamiltonian (4) is not in general self-adjoint (in conventional physics parlance, is not Hermitian). Self-adjoint 4 parameter extensions have been derived in [16], such extension admits Hamiltonian (4) as one of its members [7,16–19]. Let us emphasize that Hamiltonian $H_D$ together with the matching condition $\varphi(x)|_{x=0}=0$ is self-adjoint.

We think the misconception in the Morales *et. al.* paper could have arisen from their ideas on how the one-dimensional hydrogen atom problem come to be. As they say that, according to certain authors [1], its equation arises from the radial Schrödinger equation of the (3D) hydrogen atom merely by substituting $r$ by $x$ and a vanishing angular momentum $l = 0$. Given such assertion, we assume that they think the 1D hydrogen atom is a purely formal problem with little or no relation to any actual systems. This, however, is not so. There are specific problems which lead to essentially one-dimensional quantum motions which may be described by Hamiltonian (4). Examples of such problems are an hydrogen atom placed in a constant but super-strong magnetic field **B** [20–22], or the problem of the motion of an electron sitting on a surface producing an image charge as happens to electrons over a pool of liquid helium. In this last case, given the charge and its image is hence clear that the electron is acted by a Coulomb interaction [23]. In the case of the hydrogen atom within a **B** field, any electron state may be expressed as a product of transverse Landau states times a state depending on a coordinate parallel to **B** — states with no spherical symmetry [21]. The motion tranverse to the magnetic field is classically restrained to distances of the order of $\rho_c = (c/\mathbf{B})^{1/2}$. In the quantum case $\rho_c$ may be called the mean size of the Landau states. So, as the intensity of the magnetic field is increased, $\rho_c \to 0$ leaving only the motion along **B** for a dynamical description [20]. When the ($x$-pointing) magnetic field is super-strong the potential felt by the electron can be approximated as

$$V(r) \approx \lim_{B\to\infty} -\frac{1}{\sqrt{\rho_c^2 + x^2}} - \frac{1}{|x|} \qquad (5)$$

This is the potential used in Equation (4). Hence the name one-dimensional hydrogen atom is justified: it is just an hydrogen atom constrained to move in one direction and under the assumption that any transverse motions can be disregarded for field strengths $B \sim 10^9$ Gauss typical of neutron stars [24] they are certainly very small. It is worth noting that an hydrogen atom in a magnetic field has two integrable cases: 1) when B=0, and, 2) when B=∞.

As we have shown previously [3–6], the two eigenfunctions describing the ground state of the one dimensional hydrogen atom are

$$\psi_0^+(x) = \begin{cases} 2xL_0^1(2x)\exp(-x) & if \ \ x \geq 0 \\ 0 & if \ \ x < 0 \end{cases} \qquad (6)$$

and

$$\psi_0^-(x) = \begin{cases} 0 & if \ \ x > 0 \\ 2xL_0^1(-2x)\exp(x) & if \ \ x \leq 0 \end{cases} \qquad (7)$$

where the $L_0^1(x)$ are generalized Laguerre polynomials [17]. Notice the vanishing of the eigenfunctions at $x = 0$ and the explicit separation between the $x > 0$ and the $x < 0$ regions. This is one of the manifestations of the superselection rule which, among other things, prohibits any superposition of the right $\psi_0^+$ with the left $\psi_0^-$ eigenstates. The energy eigenstates of the problem are given by a Balmer-like formula [4,13,25,26] $E_n = -1/2 \ n^2$, $n=1,2,3, \ldots$, so the ground state energy is $E_1 = -1/2$.

With the ground eigenstates given above, the superpotential can be easily calculated as [3,27,28]

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)} = \text{sgn}(x) - \frac{1}{x} \qquad (8)$$

where $\text{sgn}(x)$ is the signum function and we have included in a single formula the consequences of both the right and the left eigenfunctions. Using the superpotential, the corresponding partner potentials are readily evaluated

$$V+(x) = \frac{1}{2} - \frac{1}{|x|} + \frac{1}{x^2}, \ and \ V-(x) = -\frac{1}{|x|} + \frac{1}{2} \qquad (9)$$

where, clearly, $V_-$ is the one-dimensional hydrogen atom potential, $V_{1DH}$, but shifted so that its ground state energy is zero, and $V_+$ is the partner potential. Also, the raising and lowering operators are

$$A^+ = -\frac{d}{dx} + W \qquad (10)$$

and

$$A = \frac{d}{dx} - W \qquad (11)$$

where, as it is easy to show,

$$[A, A^+] = 2\frac{dW}{dx} \qquad (12)$$

and

$$V_+ + V_- = 2W^2 \qquad (13)$$

The results (8) to (13) establish that the one dimensional potential $V_{\text{1DH}}$ can be regarded as stemming from the superpotential $W(x)$ in Equation (8). In [4,29], we have discussed a complete supersymmetric extension of the one-dimensional hydrogen atom problem, with Hamiltonian

$$H_{susy} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2x^2} - \frac{1}{|x|} + \frac{1}{2} + \frac{1}{2x^2}\sigma_z \qquad (14)$$

where $\sigma_z$ is a standard Pauli matrix which is needed to operate on both the the fermionic and bosonic sectors of the system. But, as the motivations of [29] were the similarities between light-cone singularities in quantum field theory with the singularity in (4), the results in [29] are not all related to the present discussion. Second, that Morales *et al.* have mistaken the paper they cite (reference [21] in their paper, reference [30] in this work) for other of our papers dealing with the one-dimensional hydrogen atom, since [30] has nothing to do with the problem at hand. It deals with a solvable model in relativistic quantum mechanics, the Dirac oscillator, which at the time was thought to have applications in QCD. They should have cited [3,5,29] instead.

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