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Weighted (λ,μ)-Ideal Statistical Convergence and Strongly Weighted (λ,μ)-Ideal Convergence of Double Sequences of Fuzzy Numbers

Xue Feng

College of Mathematics and Statistics, Qinghai Minzu University, Xining, China
Email: fengx8023@163.com

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Abstract

The paper aims to investigate different types of weighted ideal statistical convergence and strongly weighted ideal convergence of double sequences of fuzzy numbers. Relations connecting ideal statistical convergence and strongly ideal convergence have been investigated in the environment of the newly defined classes of double sequences of fuzzy numbers. At the same time, we have examined relevant inclusion relations concerning weighted (λ,μ)-ideal statistical convergence and strongly weighted (λ,μ)-ideal convergence of double sequences of fuzzy numbers. Also, some properties of these new sequence spaces are investigated.

Keywords

Fuzzy Numbers, Ideal Statistical Convergence, Double Sequences of Fuzzy Numbers, Weighted

1. Introduction

In 1965, Zadeh [1], an expert in cybernetics at University of California, first proposed the concept of fuzzy set theory. Since its inception, fuzzy set theory and its applications have been attracting the attention of researchers from various areas of science, engineering and technology. In daily life, the practical problems we have to solve often involve uncertainty, which can be expressed by fuzzy number [2]. Therefore, in the following research work, the convergence problem of sequences of fuzzy numbers is particularly important. The concept of statistical convergence of fuzzy sequence is defined by Savas [3], at the same time, statistical convergence of sequences of fuzzy numbers is expressed by the sequences of fuzzy numbers with zero natural density and the general convergent sequences of fuzzy numbers. In 1986, Matloka [4] introduced the concepts of bounded and convergent sequences of fuzzy numbers and studied their properties. In 1989, Nanda [5] studied the bounded and convergent spaces of fuzzy numbers and established that they are complete metric spaces. In 1995, Naray and Savas [6] extended the concept of statistical convergence to sequences of fuzzy numbers and showed that a sequence of fuzzy numbers is statistically convergent if and only if it is statistically Cauchy. In recent years, the problem of statistical convergence of sequences of fuzzy numbers has been studied extensively by Talo [7], Balen [8], Cinar [9] and Dutta [10], some interesting results related to statistical convergence of sequences of fuzzy numbers and related notions can also be found.

In this paper, we give the concept of weighted (λ,μ)-ideal statistical convergence and strongly weighted (λ,μ)-ideal convergence of double sequences of fuzzy numbers. And we have examined relevant inclusion relations concerning different types of weight ideal statistical convergence and strongly weight ideal convergence of double sequences of fuzzy numbers.

2. Definitions and Preliminaries

In this section, we give some basic notions which will be used throughout the paper.

Let $\tilde{A}\in\tilde{F}(R)$ be a fuzzy subset on *R*. If $\tilde{A}$ is convex, normal, upper semicontinuous and has compact support, we say that $\tilde{A}$ is a fuzzy number [11] [12] [13]. Let $\tilde{R}^c$ denote the set of all fuzzy numbers.

For $\tilde{A}\in\tilde{R}^c$, we write the level set of $\tilde{A}$ as $A_\lambda=\{x:A(x)\geq\lambda\}$ and $A_\lambda=\left[A_\lambda^-,A_\lambda^+\right]$. Let $\tilde{A},\tilde{B}\in\tilde{R}^c$, we define $\tilde{A}+\tilde{B}=\tilde{C}$ iff $A_\lambda+B_\lambda=C_\lambda$, $\lambda\in[0,1]$ iff $A_\lambda^-+B_\lambda^-=C_\lambda^-$ and $A_\lambda^++B_\lambda^+=C_\lambda^+$ for any $\lambda\in[0,1]$. $A_\lambda\cdot B_\lambda=C_\lambda$, where

$$C_\lambda^-=\min\left\{A_\lambda^-\cdot B_\lambda^-,A_\lambda^-\cdot B_\lambda^+,A_\lambda^+\cdot B_\lambda^-,A_\lambda^+\cdot B_\lambda^+\right\},\tag{2-1}$$

$$C_\lambda^+=\max\left\{A_\lambda^-\cdot B_\lambda^-,A_\lambda^-\cdot B_\lambda^+,A_\lambda^+\cdot B_\lambda^-,A_\lambda^+\cdot B_\lambda^+\right\}.\tag{2-2}$$

Define

$$D(\tilde{A},\tilde{B})=\sup_{\lambda\in[0,1]}d(A_\lambda,B_\lambda)=\sup_{\lambda\in[0,1]}\max\left\{\left|A_\lambda^--B_\lambda^-\right|,\left|A_\lambda^+-B_\lambda^+\right|\right\},\tag{2-3}$$

where d is the Hausdorff metric. $D(\tilde{A},\tilde{B})$ is called the distance between $\tilde{A}$ and $\tilde{B}$.

Using the results of [11] [12] [13], we see that

1) $(\tilde{R}^c,D)$ is a complete metric space,
2) $D(u+w,v+w)=D(u,v)$,
3) $D(ku,kv)=|k|D(u,v), k\in R$,
4) $D(u+v,w+e)\leq D(u,w)+D(v,e)$,

5) $D(u+v,\bar{0}) \le D(u,\bar{0}) + D(v,\bar{0})$,

6) $D(u+v,w) \le D(u,w) + D(v+\bar{0})$,

Where $u,v,w,e \in \tilde{R}^c$, $\tilde{0}$ represents zero fuzzy number.

Let X is a nonempty set, $I \subset 2^X$ is said to be ideal on X [14] [15], if:

1) $\varnothing \in I$;

2) if $A,B \in I$, then $A \cup B \in I$;

3) For $A \in I$, if $B \subset A$, then $B \in I$.

Especially, if $I \ne \varnothing$ and $X \not\subset I$, then I is said to be a nontrivial ideal on X.

A sequence $\{x_n\}$ of fuzzy numbers is said to be statistically convergent to a fuzzy number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : D(x_n, x_0) \ge \varepsilon\}$ has natural density zero. The fuzzy number x_0 is called the statistical limit of the sequence $\{x_n\}$ and we write $st - \lim_{n\to\infty} x_n = x_0$. A sequence $\{x_n\}$ of fuzzy numbers is said to be ideal statistically convergent to a fuzzy number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{k \le n : D(x_n, x_0) \ge \varepsilon\} \in I$, where I ia a nontrivial ideal on X [16] [17].

A double sequence of fuzzy numbers $x = \{x_{jk}\}$ is said to be bounded if there exists a positive number M such that $D(x_{jk}, \bar{0}) < M$ for all $j,k \in N$, *i.e.* if $\sup_{j,k\in N} D(x_{jk}, \bar{0}) < \infty$, where $N = \{0,1,2,\cdots\}$ [14].

Let $K \subseteq N \times N$ and $K(m,n) = \{(j,k) : j \le m, k \le n : m,n \in K\}$. The number $\delta_2(k) = P - \lim_{m,n} \frac{1}{mn} |K(m,n)|$ is called the double natural density of K, provided the limit exists [18] [19].

A double sequence of fuzzy numbers $x = \{x_{jk}\}$ is said to be statistically convergent to $L \in E^1$ if for every $\varepsilon > 0$, $\delta_2(K(m,n)) = 0$, where $K(m,n) = \{(j,k) : j \le m, k \le n : D(x_{jk}, L) \ge \varepsilon\}$, *i.e.*,

$$P - \lim_{m,n} \frac{1}{mn} \left| \{(j,k) : j \le m, k \le n : D(x_{jk}, L) \ge \varepsilon\} \right| = 0.$$

In this case, we write, $st_2 - \lim x = L$. The set of all double statistically convergent sequences of fuzzy numbers is denoted by $st^2(F)$ [20] [21].

3. Main Results

Definition 3.1. Let $p := \{p_j\}_{j=0}^{\infty}$ and $q := \{q_k\}_{k=0}^{\infty}$ be sequences of nonnegative numbers such that $p_m \ge 0$, $m = 1,2,3,\cdots$, $p_0 > 0$ and $q_n \ge n$, $n = 1,2,3,\cdots$, $q_0 > 0$ with

$$P_m = \sum_{j=0}^{m} p_j \to \infty, \text{ as } m \to \infty.$$

$$Q_n = \sum_{k=0}^{n} q_k \to \infty, \text{ as } n \to \infty.$$

The weighted mean $t_{mn}^{\beta\gamma}$ is defined as

$$t_{mn}^{11} = \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k x_{jk},$$

$$t_{mn}^{10} = \frac{1}{P_m}\sum_{j=0}^{m} p_j x_{jn},$$

$$t_{mn}^{01} = \frac{1}{Q_m}\sum_{k=0}^{n} q_k x_{mk},$$

where $m,n \ge 0$ and $(\beta,\gamma) = (1,1),(1,0),(0,1)$.

Definition 3.2. A double sequence of fuzzy numbers $x = \{x_{jk}\}$ is weighted ideal statistically convergent to x_0 if for every $\varepsilon > 0$, $\delta > 0$, we have

$$\left\{ m,n \in N \times N : \frac{1}{P_m Q_n} \left| \left\{ (j,k) : j \le P_m, k \le Q_n : p_j q_k D(x_{jk}, x_0) \ge \varepsilon \right\} \right| \ge \delta \right\} \in I.$$

In this case, we write, $x_{jk} \to x_0 \left(S_{\bar{N}_2} \right)$.

Where let $k \in N \times N$. We define the double weighted density of K by

$$\delta_{\bar{N}_2}(K) =: \lim_{n,m} \frac{1}{P_m Q_n} \left| K_{P_m Q_n}(m,n) \right|$$

where $K_{P_m Q_n}(m,n) =: \left\{ (j,k) : j \le P_m, k \le Q_n : p_j q_k D(x_{jk}, x_0) \ge \varepsilon \right\}$,
$\liminf p_n > 0$, $\liminf q_m > 0$.

Definition 3.3. A double sequence of fuzzy numbers $x = \{x_{jk}\}$ is strongly weight ideal convergent to x_0 if

$$\left\{ m,n \in N \times N : \left| \left\{ (j,k) : \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k D(x_{jk}, x_0) \ge \varepsilon \right\} \right| \ge \delta \right\} \in I.$$

and we write $x_{jk} \to x_0 \left(W_{\bar{N}_2} \right)$.

Definition 3.4. Let $\lambda = \{\lambda_m\}$ and $\mu = \{\mu_n\}$ be two nondecreasing sequence of positive real numbers such that each tending to ∞ and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$; $\mu_{n+1} \le \mu_n + 1$, $\mu_1 = 1$.

Let $p = \{p_j\}$ and $q = \{q_k\}$ be two sequence of nonnegative real numbers such that $p_m \ge 0$, $m = 1,2,3,\cdots$, $p_0 > 0$ and $q_n \ge 0$, $n = 1,2,3,\cdots$, $q_0 > 0$ with

$$P_{\lambda_m} = \sum_{j=J_m} p_j \to \infty, \text{ as } m \to \infty.$$

$$Q_{\mu_n} = \sum_{k=I_n} q_k \to \infty, \text{ as } n \to \infty.$$

where $J_m = [m - \lambda_m + 1, m]$, $I_n = [n - \mu_n + 1, n]$.

We define generalized weighted mean as follows:

$$\sigma_{mn}^{11} = \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{j=J_m} \sum_{k=I_n} p_j q_k x_{jk},$$

$$\sigma_{mn}^{10} = \frac{1}{P_{\lambda_m}} \sum_{j=J_m} p_j x_{jn},$$

$$\sigma_{mn}^{01} = \frac{1}{Q_{\mu_n}} \sum_{k=I_n} q_k x_{mk}.$$

Definition 3.5. A double sequence of fuzzy numbers $x = \{x_{jk}\}$ is said to be weighted (λ,μ)-ideal statistically convergent to x_0 if for every $\varepsilon > 0$, $\delta > 0$, we have

$$\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(x_{jk},x_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.$$

In this case, we write $x_{jk}\to x_0\left(S_{\bar{N}(\lambda,\mu)}\right)$. We denote the set of all weight (λ,μ)-ideal statistically convergent double sequences of fuzzy numbers by $S_{\bar{N}(\lambda,\mu)}$.

Definition 3.6. A double sequence of fuzzy numbers $x=\{x_{jk}\}$ is said to be strongly weight (λ,μ)-ideal convergent to x_0 if

$$\left\{m,n\in N\times N:\left|\left\{(j,k):\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_kD(x_{jk},x_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.$$

In this case, we write $x_{jk}\to x_0\left(W_{\bar{N}(\lambda,\mu)}\right)$.

Remark 3.7. When we take $\lambda_m=m,\mu_n=n$ for all $m,n\in N$, weighted (λ,μ)-ideal statistically convergence reduces to weighted ideal statistically convergence; strongly weight (λ,μ)-ideal convergence reduces to strongly weight ideal convergence.

Remark 3.8. When we take $p_j=1,q_k=1$ for all $j,k\in N$ and $\lambda_m=m,\mu_n=n$ for all $m,n\in N$, weighted (λ,μ)-ideal statistically convergence reduces to ideal statistically convergence; strongly weight (λ,μ)-ideal convergence reduces to strongly ideal convergence.

Theorem 3.9. Let $x=\{x_{jk}\},y=\{y_{jk}\}$ are the sequence of fuzzy numbers:

1) If $x_{jk}\to x_0\left(S_{\bar{N}(\lambda,\mu)}\right)$ and $c\in R$, then $cx_{jk}\to cx_0\left(S_{\bar{N}(\lambda,\mu)}\right)$;

2) If $x_{jk}\to x_0\left(S_{\bar{N}(\lambda,\mu)}\right),y_{jk}\to y_0\left(S_{\bar{N}(\lambda,\mu)}\right)$ then $x_{jk}+y_{jk}\to x_0+y_0\left(S_{\bar{N}(\lambda,\mu)}\right)$.

Proof. 1) When $c=0$, the conclusion is clearly established.

Let $c\neq 0$, we have

$$\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(cx_{jk},cx_0)\ge\varepsilon\right\}\right|$$
$$\le\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(x_{jk},x_0)\ge\frac{\varepsilon}{c}\right\}\right|$$

So

$$\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(cx_{jk},cx_0)\ge\varepsilon\right\}\right|\ge\delta\right\}$$
$$\subset\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(x_{jk},x_0)\ge\frac{\varepsilon}{c}\right\}\right|\ge\delta\right\}\in I.$$

We have $cx_{jk}\to cx_0\left(S_{\bar{N}(\lambda,\mu)}\right)$.

2) Let $x_{jk}\to x_0\left(S_{\bar{N}(\lambda,\mu)}\right),y_{jk}\to y_0\left(S_{\bar{N}(\lambda,\mu)}\right)$, then

$$\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(x_{jk},x_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I;$$

$$\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(Y_{jk},Y_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.$$

On the other hand,

$$\begin{aligned}D\left(x_{jk}+y_{jk},x_0+y_0\right)&\le D\left(x_{jk}+y_{jk},x_0+y_{jk}\right)+D\left(x_0+y_{jk},x_0+y_0\right)\\&=D\left(x_{jk},x_0\right)+D\left(y_{jk},y_0\right).\end{aligned}$$

for $\forall\varepsilon>0$, we have

$$\begin{aligned}&\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(x_{jk}+y_{jk},x_0+y_0\right)\ge\varepsilon\right\}\right|\\&\le\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(x_{jk},x_0\right)\ge\frac{\varepsilon}{2}\right\}\right|\\&+\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(y_{jk},y_0\right)\ge\frac{\varepsilon}{2}\right\}\right|.\end{aligned}$$

So

$$\begin{aligned}&\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(x_{jk}+y_{jk},x_0+y_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\&\subseteq\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(x_{jk},x_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\&\cup\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(y_{jk},y_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.\end{aligned}$$

We can get $x_{jk}+y_{jk}\to x_0+y_0\left(S_{\bar{N}(\lambda,\mu)}\right)$.

In case $\lambda_m=m,\mu_n=n$ for all $m,n\in N$, $S_{\bar{N}(\lambda,\mu)}$-ideal statistical convergence reduces to $S_{\bar{N}_2}$-ideal statistical convergence and then we have the following corollary.

Corollary 3.10. Let $x=\left\{x_{jk}\right\},y=\left\{y_{jk}\right\}$ are the sequence of fuzzy numbers:

1) If $x_{jk}\to x_0\left(S_{\bar{N}_2}\right)$ and $c\in R$, then $cx_{jk}\to cx_0\left(S_{\bar{N}_2}\right)$;

2) If $x_{jk}\to x_0\left(S_{\bar{N}_2}\right),y_{jk}\to y_0\left(S_{\bar{N}_2}\right)$ then $x_{jk}+y_{jk}\to x_0+y_0\left(S_{\bar{N}_2}\right)$.

Theorem 3.11. Let $x=\left\{x_{jk}\right\}$ is the sequence of fuzzy number, there is a $S_{\bar{N}(\lambda,\mu)}$-ideal statistically convergent sequence of fuzzy number $y=\left\{y_{jk}\right\}$, such that $\left\{x_{jk}\right\}=\left\{y_{jk}\right\}$ for almost all j,k, then $y=\left\{y_{jk}\right\}$ also $S_{\bar{N}(\lambda,\mu)}$-ideal statistical convergence.

Proof. For almost all j,k, we have $\left\{x_{jk}\right\}=\left\{y_{jk}\right\}$, and $y_{jk}\to y_0\left(S_{\bar{N}(\lambda,\mu)}\right)$. Let $\varepsilon>0$, $\delta>0$, then

$$\begin{aligned}&\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD\left(x_{jk},x_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\&\subseteq\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:p_jq_kD(y_{jk},y_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\\&\cup\left\{(j,k):j\le P_{\lambda_m},k\le Q_{\mu_n}:x_{jk}\ne y_{jk}\right\}.\end{aligned}$$

Let $S = S(\varepsilon)$ is the number of elements in the set of $\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : x_{jk} \ne y_{jk}\}$, then

$$\left|\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(x_{jk}, x_0) \ge \varepsilon\}\right|$$
$$\le \left|\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(y_{jk}, y_0) \ge \varepsilon\}\right| + S.$$

So

$$\left\{m,n \in N \times N : \frac{1}{P_{\lambda_m} Q_{\mu_n}} \left|\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(x_{jk}, x_0) \ge \varepsilon\}\right| \ge \delta\right\} \in I.$$

The theorem proved.

In case $\lambda_m = m, \mu_n = n$ for all $m,n \in N$, $S_{\bar{N}_{(\lambda,\mu)}}$-ideal statistical convergence reduces to $S_{\bar{N}_2}$-ideal statistical convergence and then we have the following corollary.

Corollary 3.12. Let $x = \{x_{jk}\}$ is the sequence of fuzzy number, there is a $S_{\bar{N}_2}$-ideal statistically convergent sequence of fuzzy number $y = \{y_{jk}\}$, such that $\{x_{jk}\} = \{y_{jk}\}$ for almost all j,k, then $y = \{y_{jk}\}$ also $S_{\bar{N}_2}$-ideal statistical convergence.

Theorem 3.13. Let $p_j q_k D(x_{jk}, x_0) \le M$ for all $j,k \in N$. If a double sequence of fuzzy numbers $x = \{x_{jk}\}$ is weight (λ,μ)-ideal statistically convergent to x_0 then it is strongly weight (λ,μ)-ideal convergent to x_0.

Proof. Suppose $p_j q_k D(x_{jk}, x_0) \le M$ for all $j,k \in N$ and the double sequence of fuzzy numbers $x = \{x_{jk}\}$ is weight (λ,μ)-ideal statistically convergent to x_0. We note

$$K_{P_{\lambda_m} Q_{\mu_n}}(\varepsilon) = \{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(x_{jk}, x_0) \ge \varepsilon\}$$

$$\begin{aligned}
&\sum_{j \in J_m} \sum_{k \in I_n} p_j q_k D(x_{jk}, x_0) \\
&= \sum_{j \in J_m} \sum_{k \in I_n, k \in K_{P_{\lambda_m} Q_{\mu_n}}} p_j q_k D(x_{jk}, x_0) + \sum_{j \in J_m} \sum_{k \in I_n, k \in K^C_{P_{\lambda_m} Q_{\mu_n}}} p_j q_k D(x_{jk}, x_0) \\
&> \sum_{j \in J_m} \sum_{k \in I_n, k \in K_{P_{\lambda_m} Q_{\mu_n}}} p_j q_k D(x_{jk}, x_0) \\
&= \left|\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(x_{jk}, x_0) \ge \varepsilon\}\right| \cdot M.
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\{m,n \in N \times N : \left|\left\{(j,k): \frac{1}{P_{\lambda_m} Q_{\mu_n}} \sum_{j \in J_m} \sum_{k \in I_n} p_j q_k D(x_{jk}, x_0) \ge \varepsilon\right\}\right| \ge \delta\right\} \\
&\subset \left\{m,n \in N \times N : \frac{1}{P_{\lambda_m} Q_{\mu_n}} \left|\{(j,k): j \le P_{\lambda_m}, k \le Q_{\mu_n} : p_j q_k D(x_{jk}, x_0) \ge \varepsilon\}\right| \ge \delta\right\} \in I.
\end{aligned}$$

i.e. $W_{\bar{N}_{(\lambda,\mu)}} \subset S_{\bar{N}_{(\lambda,\mu)}}$.

Theorem 3.14. Let a double sequence of fuzzy numbers x_{jk} is strongly weighted (λ,μ)-ideal convergent to x_0, then x_{jk} is weighted (λ,μ)-ideal statistically convergent to x_0.

Proof. Let $K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)=\left\{(j,k): j\le P_{\lambda_m}, k\le Q_{\mu_n} : p_j q_k D(x_{jk},x_0)\ge\varepsilon\right\}$, then

$$\begin{aligned}&\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_kD(x_{jk},x_0)\\&=\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n,k\in K_{P_{\lambda_m}Q_{\mu_n}}}p_jq_kD(x_{jk},x_0)\\&\quad+\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n,k\in K^C_{P_{\lambda_m}Q_{\mu_n}}}p_jq_kD(x_{jk},x_0)\\&>\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n,k\in K_{P_{\lambda_m}Q_{\mu_n}}}p_jq_kD(x_{jk},x_0)\\&\ge\frac{\varepsilon}{P_{\lambda_m}Q_{\mu_n}}\left|K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)\right|.\end{aligned}$$

where $K_{P_{\lambda_m}Q_{\mu_n}}(\varepsilon)=\left\{(j,k): j\le P_{\lambda_m}, k\le Q_{\mu_n} : p_j q_k D(x_{jk},x_0)\ge\varepsilon\right\}$.

We have

$$\begin{aligned}&\left\{m,n\in N\times N:\frac{1}{P_{\lambda_m}Q_{\mu_n}}\left|\left\{(j,k): j\le P_{\lambda_m}, k\le Q_{\mu_n} : p_jq_kD(x_{jk},x_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\\&\subset\left\{m,n\in N\times N:\left|\left\{(j,k):\frac{1}{P_{\lambda_m}Q_{\mu_n}}\sum_{j\in J_m}\sum_{k\in I_n}p_jq_kD(x_{jk},x_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.\end{aligned}$$

We get x_{jk} is weighted (λ,μ)-ideal statistically convergent to x_0.

4. Conclusion

In this article, we aim to investigate different types of weighted ideal statistical convergence and strongly weighted ideal convergence of double sequences of fuzzy numbers. Relations connecting ideal statistical convergence and strongly ideal convergence have been investigated in the environment of the newly defined classes of double sequences of fuzzy numbers. At the same time, we have examined relevant inclusion relations concerning weighted (λ,μ)-ideal statistical convergence and strongly weighted (λ,μ)-ideal convergence of double sequences of fuzzy numbers.

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Conflicts of Interest

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