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Existence and Stability of Standing Waves for the Nonlinear Schrödinger Equation with Combined Nonlinearities and a Partial Harmonic Potential

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Abstract

In this paper, we study the existence of standing waves for the nonlinear Schrödinger equation with combined power-type nonlinearities and a partial harmonic potential. In the L^2-supercritical case, we obtain the existence and stability of standing waves. Our results are complements to the results of Carles and Il'yasov's artical, where orbital stability of standing waves have been studied for the 2D Schrödinger equation with combined nonlinearities and harmonic potential.

Keywords

Nonlinear Schrödinger Equation, Orbital Stability, Standing Waves

1. Introduction

In this paper, we study the existence and stability of standing waves for the nonlinear Schrödinger equation with combined power-type nonlinearities and a partial harmonic potential

$$\begin{cases} i\partial_t \psi + \Delta \psi - W(x)\psi + \mu |\psi|^p \psi + |\psi|^q \psi = 0, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1}$$

where ψ is a complex-valued function of (t,x), $\mu > 0$, $N \geq 3$, $0 < p < \frac{4}{N} < q < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$. The external potential W describes the electromagnetic trap for the condensate and is usually chosen to be an istropic quadratic confinement, *i.e.*,

$$W(x)=\sum_{i=1}^{N}a_i^2x_i^2,\ a_i\in\mathbb{R},$$

where $a_1,a_2,\cdots,a_N\in\mathbb{R}$ represent the corresponding trap frequency in each spatial direction. The Gross-Pitaevskii equation with N particle bodies are strictly derived by Gross and Pitaevskii, see [1] [2]. Due to the inclusion of a quadratic potential, the natural energy space for studying Equation (1.1) is given by the following expression

$$X:=\left\{u\in H^1\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}W(x)|u(x)|^2\,\mathrm{d}x<\infty\right\},$$

with the norm

$$\|u\|_X^2=\|\nabla u\|_{L^2}^2+\|u\|_{L^2}^2+\int_{\mathbb{R}^N}W(x)|u(x)|^2\,\mathrm{d}x.$$

Different forms of the potential W correspond to different physical meanings. In the case $W=0$, Equation (1.1) arises in various areas of physics and mathematics. The typical class of nonlinear dispersion equation had been proposed by Schrödinger in [3]. When $W=\sum_{i=1}^{N}a_i^2x_i^2$, $a_i\in\mathbb{R}\setminus\{0\}$, it is a nonlinear Schrödinger equation with harmonic limiting potential and W indicates that the external potential is uniformly distributed in all directions of space in a harmonic form. This type of equations has been studied in [4] [5] [6]. In fact, when $W(x)=\sum_{i=1}^{N}a_i^2x_i^2$, $a_i\in\mathbb{R}$, some coefficients of $W(x)$ will vanish, the harmonic potential becomes partial harmonic potential. Then Equation (1.1) does not keep invariant by translation. The orbital stability of standing waves for the inhomogeneous Gross-Pitaevskii equation has been studied in [7].

In [4], Carles and Il'yasov considered the nonlinear Schrödinger equation with a harmonic potential in the presence of two combined energy-subcritical power nonlinearities. They address the equations of the existence and the orbital stability of the set of standing waves by the method of fundamental frequency solutions. This method makes it possible to describe accurately the set of fundamental frequency standing waves and ground states, and to prove its orbital stability. On this basis, this paper converts the harmonic potential into a partial harmonic potential. At this time, the compactness disappears, which makes it more difficult to discuss the stability of the standing waves for the equation. Therefore, we must seek new ways to address the issues raised in this article.

Equation (1.1) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $\psi(t,x)=\mathrm{e}^{i\omega t}u(x)$, where $\omega\in\mathbb{R}$, and the function $u\in X$ solves the following elliptic equation

$$-\Delta u+\omega u+W(x)u-\mu|u|^p u-|u|^q u=0. \tag{1.2}$$

A possible choice is then to fix $w\in\mathbb{R}$, and to search for solutions to (1.2) as critical points of action functional

$$\mathcal{A}_{\mu,\omega}(u):=E_\mu(u)+\frac{\omega}{2}\|u\|_{L^2}^2, \tag{1.3}$$

where the energy $E_\mu(u)$ is defined as

$$E_{\mu}(u)=\frac{1}{2}\|\nabla u\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}^N}W(x)|u(x)|^2\,\mathrm{d}x-\frac{\mu}{p+2}\|u\|_{L^{p+2}}^{p+2}-\frac{1}{q+2}\|u\|_{L^{q+2}}^{q+2}. \quad (1.4)$$

For the Equation (1.1), an important issue is to consider the stability of standing waves, which is defined as follows:

Definition 1.1. *A set* $\mathcal{M}$ *is orbitally stable if for any given* $\varepsilon>0$, *there exists* $\delta>0$ *such that for any initial data* ψ_0 *satisfying*

$$\inf_{u\in\mathcal{M}}\|\psi_0-u\|_X<\delta,$$

the corresponding solution $\psi(t)$ *of (1.1) satisfies*

$$\inf_{u\in\mathcal{M}}\|\psi(t)-u\|_X<\varepsilon, \forall\, t\geq 0.$$

Given this definition, for the sake of stability, we require that the solution of (1.1) exists globally, at least for initial data ψ_0 close enough to $\mathcal{M}$. In L^2-supercritical case, according to the local well-posedness theory of NLS, small initial data NLS solution exists globally, while for some large initial data, the solution may blow up in finite time. Therefore, it is especially important to pay attention to whether there is a stable standing waves in this case.

In order to study the orbital stability of standing waves, we apply the idea by Cazenave and Lions in [8], consider the following constrained minimization problem

$$m(c):=\inf_{u\in S(c)}E_{\mu}(u),$$

where

$$S(c)=\left\{u\in X:\|u\|_{L^2}=c\right\}.$$

However, since the nonlinearities is the L^2-supercritical, the energy functional is unbounded from below on $S(c)$. Indeed, when $0<p<\frac{4}{N}<q<\frac{4}{N-2}$, taking $u\in X$ such that $\|u\|_{L^2}=c$, then we have

$$\begin{aligned}E_{\mu}(u^{\lambda})&=\frac{1}{2}\|\nabla u^{\lambda}\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}^N}W(x)|u^{\lambda}(x)|^2\,\mathrm{d}x-\frac{\mu}{p+2}\|u^{\lambda}\|_{L^{p+2}}^{p+2}-\frac{1}{q+2}\|u^{\lambda}\|_{L^{q+2}}^{q+2}\\&=\frac{\lambda^2}{2}\|\nabla u\|_{L^2}^2+\frac{\lambda^{-2}}{2}\int_{\mathbb{R}^N}W(x)|u(x)|^2\,\mathrm{d}x-\frac{\mu\lambda^{\frac{Np}{2}}}{p+2}\|u\|_{L^{p+2}}^{p+2}-\frac{\lambda^{\frac{Nq}{2}}}{q+2}\|u\|_{L^{q+2}}^{q+2} \quad (1.5)\\&\to-\infty,\end{aligned}$$

as $\lambda\to\infty$. Therefore, we cannot study the existence and stability of standing waves of Equation (1.1) by considering the global minimization problem. Due to this type of problems has been considered in [9] [10] [11] by studying the corresponding local minimization problems, we consider the following local minimization problem: for any given $r>0$, defining

$$m(c,r):=\inf_{u\in S(c)\cap B(r)}E_{\mu}(u), \quad (1.6)$$

where

$$B(r):=\left\{u\in X:\|u\|_{\tilde{X}}\leq r\right\},$$

and $\|\cdot\|_{\tilde{X}}$ is given by

$$\|u\|_{\tilde{X}}^2 := \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^N} W(x)|u(x)|^2 \, \mathrm{d}x. \tag{1.7}$$

It can be proved that for any given $r > r_0$ with some $r_0 > 0$, there exists a $c_r > 0$, such that $S(c) \cap B(r) \neq \varnothing, \forall c < c_r$. Thus we can obtain the existence of a minimizer of $m(c,r)$. Denote the set of all minimizers of $m(c,r)$ by

$$\mathcal{M}_r(c) := \{u \in S(c) \cap B(r) : E_\mu(u) = m(c,r)\}.$$

To prove the existence and stability, the key is to show that any minimizing sequence is relatively compact. For Equation (1.1), when $a_1 a_2 \cdots a_N \neq 0$, the embedding $X \hookrightarrow L^q$ with $q \in \left[2, \frac{2N}{N-2}\right)$ is compact, the minimization problem (1.6) can be easily solved. However, when some of coefficients of $W(x)$ vanish, the embedding $X \hookrightarrow L^q$ with $q \in \left[2, \frac{2N}{N-2}\right)$ is not compact. In this case, the general method is to apply concentration compactness principle to overcome this difficulty. Then we can obtain the compactness of all minimizing sequences of (1.6) and prove the existence and stability of standing waves for (1.1). Without loss of generality, we assume

$$W(x) = \sum_{i=1}^{d} a_i^2 x_i^2, a_i \in \mathbb{R} \setminus \{0\}, \tag{1.8}$$

where $1 \leq d < N$.

According to (1.8), our main results are as follows:

Theorem 1.2. *Let* $\mu > 0$, $1 \leq d < N$, $\frac{4}{N} < q < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$, *all being fixed, then there exists* $r_0 > 0$, *such that for every given* $r > r_0$, *there exists* c_r *with* $0 < c_r < 1$, *we have for any* $c \in (0, c_r)$ *that*

1) $\varnothing \neq \mathcal{M}_r(c) \subset S(c) \cap B\left(\frac{rc}{2}\right)$;

2) *The set* $\mathcal{M}_r(c)$ *is orbitally stable.*

This paper is organized as follows: in Section 2, we given some preliminary results, which will be used later. In Section 3, we prove Theorem 1.2.

2. Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local well-posedness theory for the Cauchy problem (1.1) established in [12].

Lemma 2.1. *Let* $N \geq 3$, $1 \leq d < N$, $0 < p < \frac{4}{N} < q < \frac{4}{N-2}$, *and* $\psi_0 \in X$. *Then, there exists* $T = T(\|\psi_0\|_X)$, *such that (1.1) admits a unique solution* $\psi \in C([0,T], X)$. *Let* $[0, T^*]$ *be the maximal time interval on which the solution* ψ *is well-defined, if* $T^* < \infty$, *then* $\|\psi(t)\|_X \to \infty$ *as* $t \uparrow T^*$. *Moreover, for all* $T \in [0, T^*)$, *the solution* $\psi(t)$ *satisfies the following conservations of mass and energy*

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

and

$$E_\mu(\psi(t)) = E_\mu(\psi_0),$$

where $E_\mu(u)$ *is defined by (1.4).*

Lemma 2.2. [13] *Define*

$$\Lambda_0 = \inf_{\int_{\mathbb{R}^N}|u|^2 \mathrm{d}x=1}\left(\int_{\mathbb{R}^N}|\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} W(x)|u(x)|^2 \mathrm{d}x\right),$$

and

$$\lambda_0 = \inf_{\int_{\mathbb{R}^k}|v|^2 \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k=1}\left(\int_{\mathbb{R}^k}\left|\nabla_{x_1x_2\cdots x_k} v\right|^2 \mathrm{d}x + \int_{\mathbb{R}^k} W(x)|v|^2 \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k\right).$$

Then $\Lambda_0 = \lambda_0$.

Lemma 2.3. [14] *Let* $N \ge 1$ *and* $0 < \eta < 4/(N-2)$, *then the following sharp Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{\eta+2}}^{\eta+2} \le C_{GN}(\eta)\|u\|_{L^2}^{2+\eta(2-N)/2}\|\nabla u\|_{L^2}^{\eta N/2},$$

holds for any $u \in H^1(\mathbb{R}^N)$. *The sharp constant* $C_{GN}(\eta)$ *is*

$$C_{GN}(\eta) = \frac{2(\eta+2)}{4-(N-2)\eta}\left(\frac{4-(N-2)\eta}{N\eta}\right)^{N\eta/4}\frac{1}{\|Q\|_{L^2}^{\eta}},$$

where Q is defined in Theorem 1.4 by [8].

Lemma 2.4. [9] *Let* $0 < p < \infty$, *suppose that* $f_n \to f$ *almost everywhere and* $\{f_n\}$ *is a bounded sequence in* L^p, *then*

$$\lim_{n\to\infty}\left(\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p\right) = 0.$$

3. Proof of Theorem 1.2

In this section, we first establish a local minima structure for $E_\mu(u)$ on $S(c)$.

Lemma 3.1. *Let* $\mu > 0$, $1 \le d < N$, $0 < p < \frac{4}{N} < q < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$, *all being fixed, then there exists* $r_0 > 0$, *such that for every given* $r > r_0$, *there exists* c_r *with* $0 < c_r < 1$, *we have*

$$S(c)\cap B\left(\frac{rc}{2}\right) \ne \varnothing, \forall c > 0, \tag{3.1}$$

$$\inf_{u\in S(c)\cap B\left(\frac{rc}{2}\right)} E_\mu(u) < \inf_{u\in S(c)\cap(B(r)\setminus B(rc))} E_\mu(u), \forall c < c_r. \tag{3.2}$$

Proof. Let $u_0 \in X$ be such that $\|u_0\|_{L^2} = 2$, $\|u_0\|_{\tilde{X}}^2 = r_0^2$. Then for all $c > 0$, letting $u_c := \frac{c}{2}u_0$, we have

$$\|u_c\|_{L^2} = c,$$

and

$$\|u_c\|_{\tilde{X}}^2 = \frac{c^2 r_0^2}{4} < \frac{c^2 r^2}{4}, \forall r > r_0,$$

namely $u_c \in S(c) \cap B\left(\frac{rc}{2}\right)$. Thus (3.1) is verified.

To verify (3.2), using the Gagliardo-Nirenberg inequality, we have

$$\begin{cases} E_\mu(u) \ge \frac{1}{2}\|u\|_{\tilde{X}}^2 - \frac{\mu}{p+2} C_1 c^{p+2-\frac{Np}{2}} \|u\|_{\tilde{X}}^{\frac{Np}{2}} - \frac{1}{q+2} C_2 c^{q+2-\frac{Nq}{2}} \|u\|_{\tilde{X}}^{\frac{Nq}{2}}, \ \forall u \in S(c), \\ E_\mu(u) \le \frac{1}{2}\|u\|_{\tilde{X}}^2, \ \forall u \in S(c). \end{cases}$$

Denote $K_1 = \frac{\mu C_1}{p+2}, K_2 = \frac{C_2}{q+2}$, we define the following functions:

$$\begin{cases} f_c(t) := \frac{1}{2}t^2 - K_1 c^{p+2-\frac{Np}{2}} t^{\frac{Np}{2}} - K_2 c^{q+2-\frac{Nq}{2}} t^{\frac{Nq}{2}}, t > 0, \\ g_c(t) := \frac{1}{2}t^2. \end{cases}$$

Notice that for any $r > r_0$, there exists $0 < c_r \ll 1$, such that for all $t \in (rc, r)$ with $c < c_r$, we have

$$\begin{aligned} f_c(t) &= \frac{1}{2}t^2 - K_1 c^{p+2-\frac{Np}{2}} t^{\frac{Np}{2}} - K_2 c^{q+2-\frac{Nq}{2}} t^{\frac{Nq}{2}} \\ &= t^2 \left(\frac{1}{2} - K_1 c^{p+2-\frac{Np}{2}} t^{\frac{Np-4}{2}} - K_2 c^{q+2-\frac{Nq}{2}} t^{\frac{Nq-4}{2}} \right) \\ &\ge r^2 c^2 \left(\frac{1}{2} - K_1 c^{p+2-\frac{Np}{2}} (rc)^{\frac{Np-4}{2}} - K_2 c^{q+2-\frac{Nq}{2}} r^{\frac{Nq-4}{2}} \right) \\ &> \frac{3}{8} r^2 c^2 > g_c\left(\frac{rc}{2}\right). \end{aligned} \tag{3.3}$$

This implies that

$$g_c\left(\frac{rc}{2}\right) < \frac{3}{8} r^2 c^2 \le \inf_{t \in (rc, r)} f_c(t), \ \forall c < c_r,$$

which completes the proof.

Lemma 3.2. *Let* $\mu > 0$, $1 \le d < N$ *and* $0 < p < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$. *Let* $r > 0$ *and* $c_r > 0$ *be as Lemma 3.3. Let* $c < c_r$ *and* $\{u_n\}$ *be a minimizing sequence of (1.6). Then there exists* $\delta > 0$ *such that*

$$\lim_{n \to \infty} \int |v_n|^{p+2} \mathrm{d}x > \delta. \tag{3.4}$$

Proof. If there exists a subsequence, still denoted by $\{v_n\}$ such that $\|v_n\|_{L^{p+2}} \to 0$ as $n \to \infty$, then by the interpolation, $\|v_n\|_{L^q} \to 0$ as $n \to \infty$, for all $0 < q < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$, $v_n \in S(c) \cap B(r)$. We consequently obtain that

$$\begin{aligned} m(c,r) &= E_\mu(v_n) + o_n(1) \\ &= \frac{1}{2}\|\nabla v_n\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^N} W(x)|v_n|^2 \mathrm{d}x - \frac{\mu}{p+2}\|v_n\|_{L^{p+2}}^{p+2} - \frac{1}{q+2}\|v_n\|_{L^{q+2}}^{q+2} + o_n(1) \\ &\ge \frac{c^2}{2}\Lambda_0 + o_n(1), \end{aligned} \tag{3.5}$$

where Λ_0 is defined by Lemma 2.2.

On the other hand, since the space $\Sigma := \left\{ v \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} W(x)|v|^2 \mathrm{d}x < \infty \right\}$ is compactly embedding in $L^2(\mathbb{R}^d)$, it is standard to show that λ_0 is achieved by some $\tilde{v} \in H^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |\tilde{v}|^2 \mathrm{d}x = 1$. Let $\varphi \in H^1(\mathbb{R}^{N-d})$ satisfy $\int_{\mathbb{R}^{N-d}} |\varphi(x)|^2 \mathrm{d}x = c^2$, and set

$$u^\lambda(x) := \tilde{v}(x_1, x_2, \cdots, x_d)\varphi_\lambda(x_{d+1}, \cdots, x_N),$$

where

$$\varphi_\lambda(x_{d+1}, \cdots, x_N) := \lambda^{\frac{N-d}{2}} \varphi(\lambda x_{d+1}, \cdots, \lambda x_N).$$

Then $u^\lambda \in S(c)$ for all $\lambda > 0$. It follows that

$$\begin{aligned} E_\mu(u^\lambda) &= \frac{1}{2}\int_{\mathbb{R}^{N-d}} \left|\nabla_{x_{d+1}\cdots x_N} \varphi_\lambda\right| \mathrm{d}x_{d+1}\cdots \mathrm{d}x_N + \frac{\Lambda_0 c^2}{2} \\ &\quad - \frac{\mu}{p+2}\int_{\mathbb{R}^N} \left|\tilde{v}(x_1, x_2, \cdots, x_d)\right|^{p+2} \left|\varphi_\lambda(x_{d+1}, \cdots, x_N)\right|^{p+2} \mathrm{d}x \\ &\quad - \frac{1}{q+2}\int_{\mathbb{R}^N} \left|\tilde{v}(x_1, x_2, \cdots, x_d)\right|^{q+2} \left|\varphi_\lambda(x_{d+1}, \cdots, x_N)\right|^{q+2} \mathrm{d}x \\ &= \frac{\lambda^2}{2}\int_{\mathbb{R}^{N-d}} \left|\nabla_{x_{d+1}\cdots x_N} \varphi_\lambda\right| \mathrm{d}x_{d+1}\cdots \mathrm{d}x_N + \frac{\Lambda_0 c^2}{2} \\ &\quad - \frac{\mu\lambda^{\frac{(N-d)p}{2}}}{p+2}\int_{\mathbb{R}^N} \left|\tilde{v}(x_1, \cdots, x_d)\right|^{p+2} \left|\varphi(x_{d+1}, \cdots, x_N)\right|^{p+2} \mathrm{d}x \\ &\quad - \frac{\lambda^{\frac{(N-d)q}{2}}}{q+2}\int_{\mathbb{R}^N} \left|\tilde{v}(x_1, \cdots, x_d)\right|^{q+2} \left|\varphi(x_{d+1}, \cdots, x_N)\right|^{q+2} \mathrm{d}x, \end{aligned} \tag{3.6}$$

for $\lambda > 0$ small enough. Notice that $u^\lambda \in B(r)$ for $\lambda > 0$ sufficiently small, we consequently obtain that

$$m(c, r) \le E_\mu(u^\lambda) < \frac{1}{2}\Lambda_0 c^2.$$

This is a contradiction with (3.8). This completes the proof.

Lemma 3.3. *Let* $\mu > 0$, $1 \le d < N$ *and* $0 < p < \frac{4}{N} < q < \min\left\{\frac{4}{N-d}, \frac{4}{N-2}\right\}$. *Let* $r > 0$ *and* $c_r > 0$ *be as Lemma 3.1. Then for any* $0 < c_2 < c_1 < c_r$, *we have*

$$m(c_1, r) < m(c_2, r) + m(c_1 - c_2, r). \tag{3.7}$$

Proof. We first prove the following strict monotonicity:

$$tm(s, r) < sm(t, r), \forall\, 0 < t < s < \min\{1, c_r\}. \tag{3.8}$$

Indeed, let $v_n \subset S(t) \cap B(r)$ satisfy $E_\mu(v_n) \to m(t, r)$. Then by Lemma 3.2, there exists a $\delta_1 > 0$ such that

$$\lim_{n\to\infty} \int |v_n|^{p+2} \mathrm{d}x > \delta_1.$$

Let $u_n := \sqrt{\frac{s}{t}} v_n$, we have $u_n \in S(s) \cap B\left(\frac{rs}{2}\right) \subset S(s) \cap B(r)$. Then we get

$$
\begin{aligned}
m(s,r) &\le \lim_{n\to\infty} E_\mu(u_n) \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\|\nabla u_n(x)\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^N} W(x)|u_n|^2\,\mathrm{d}x - \frac{\mu}{p+2}\|u_n\|_{L^{p+2}}^{p+2} - \frac{1}{q+2}\|u_n\|_{L^{q+2}}^{q+2}\right] \\
&= \lim_{n\to\infty}\frac{s}{t}E_\mu(v_n) - \lim_{n\to\infty}\frac{\mu}{p+2}\left[\left(\frac{s}{t}\right)^{\frac{p+2}{2}} - \left(\frac{s}{t}\right)\right]\|v_n\|_{L^{p+2}}^{p+2} - \lim_{n\to\infty}\frac{1}{q+2}\left[\left(\frac{s}{t}\right)^{\frac{q+2}{2}} - \left(\frac{s}{t}\right)\right]\|v_n\|_{L^{q+2}}^{q+2} \\
&\le \frac{s}{t}m(t,r) - \frac{\mu}{p+2}\left[\left(\frac{s}{t}\right)^{\frac{p+2}{2}} - \left(\frac{s}{t}\right)\right]\delta_1 - \frac{1}{q+2}\left[\left(\frac{s}{t}\right)^{\frac{q+2}{2}} - \left(\frac{s}{t}\right)\right]\delta_1 + o_n(1) \\
&< \frac{s}{t}m(t,r),
\end{aligned}
\tag{3.9}
$$

which implies (3.11). Then for all $0 < c_2 < c_1 < c_r$, we get

$$
\begin{aligned}
m(c_1,r) &= \frac{c_2}{c_1}m(c_1,r) + \frac{c_1-c_2}{c_1}m(c_1,r) \\
&< \frac{c_2}{c_1}\frac{c_1}{c_2}m(c_2,r) + \frac{c_1-c_2}{c_1}\frac{c_1}{c_1-c_2}m(c_1-c_2,r) \\
&< m(c_2,r) + m(c_1-c_2,r).
\end{aligned}
\tag{3.10}
$$

Proof of Theorem 1.2. Let u_n be a minimizing sequence of $m(c,r)$, namely

$$E_\mu(u_n) \to m(c,r), \|u_n\|_{L^2} = c, \|u_n\|_{\tilde{X}} \le r.$$

Applying Lemma 3.2, there exist $\{z_n\} \subset \mathbb{R}$ and $u \in X \setminus \{0\}$, such that

$$v_n \rightharpoonup u \text{ in } X,$$

where $v_n := u_n$. We first prove $\|u\|_{L^2} = c$. If not, denote $c > c_1 := \|u\|_{L^2}$ and $c_2 := \|v_n - u\|_{L^2}$. Appling Brezis-Lieb Lemma, we have

$$\|v_n\|_{L^2} = \|v_n - u\|_{L^2} + \|u\|_{L^2} + o_n(1),$$

and

$$\|v_n\|_{\tilde{X}} = \|v_n - u\|_{\tilde{X}} + \|u\|_{\tilde{X}} + o_n(1).$$

Then we see that $u \in S(c_1) \cap B(r)$, $v_n - u \in S(c_2) \cap B(r)$. Choosing a subsequence of $\{v_n\}$ (still denote by v_n) such that $c_2 \to (c-c_1)^-$, we deduce that

$$
\begin{aligned}
m(c,r) &= E_\mu(v_n) + o_n(1) \\
&= E_\mu(v_n - u) + E_\mu(u) + o_n(1) \\
&\ge m(c_2,r) + m(c_1,r) + o_n(1) \\
&\ge \frac{c_2}{c-c_1}m(c-c_1,r) + m(c_1,r) + o_n(1) \\
&= m(c-c_1,r) + m(c_1,r) + o_n(1),
\end{aligned}
\tag{3.11}
$$

which is a contradiction with Lemma 3.3. Therefore $\|u\|_{L^2} = c$ and $v_n \to u$ strongly in L^2. Then by using the interpolation, we get that $v_n \to u$ strongly in L^s for all $s \in [2, 2^*)$. We then deduce from the weak convergence in X that $E_\mu(u) \le \liminf_{n\to\infty} E_\mu(v_n) = m(c,r)$. On the other hand, due to

$\|u\|_{\tilde{X}} \le \liminf_{n\to\infty} \|v_n\|_{\tilde{X}} \le r$, we have $E_\mu(u) \ge m(c,r)$. We consequently obtain $E_\mu(u) = m(c,r)$ and then $\lim_{n\to\infty} \|v_n\|_{\tilde{X}} = \|u\|_{\tilde{X}}$. Thus, we deduce that $v_n \to u$ strongly in *X*, namely $u_n \to u$ strongly in *X* and $u \in \mathcal{M}_r(c)$.

Next we prove that $\mathcal{M}_r(c)$ is orbitally stable by contradiction. We assume that there exist $\varepsilon_0 > 0$ and a sequence of initial data $\{\psi_{n,0}\} \subset X$ such that

$$\lim_{n\to\infty} \inf_{u\in\mathcal{M}_r(c)} \|\psi_{n,0} - u\|_X = 0, \tag{3.12}$$

and there exist a sequence $\{t_n\} \subset \mathbb{R}$ such that the maximal solution $\psi_n(t)$ with $\psi_n(0) = \psi_{n,0}$ satisfies that

$$\inf_{u\in\mathcal{M}_r(c)} \|\psi_n(t_n) - u\|_X \ge \varepsilon_0. \tag{3.13}$$

Without restriction, we can assume $\psi_{n,0} \in S(c)$ such that $\{\psi_{n,0}\}$ is a minimizing sequence of (1.6). According to Lemma 3.1, when *n* is sufficiently large, we have $\psi_{n,0} \in S(c) \cap B\left(\frac{rc}{2}\right)$, which together with $E_\mu(\psi_n(t)) = E_\mu(\psi_{n,0})$ and $\|\psi_n(t)\|_{L^2} = \|\psi_{n,0}\|_{L^2} = c$, implies that $\psi_n(t_n) \in S(c) \cap B(r)$ is a minimizing sequence for (1.6). Then we have $\psi_n(t_n) \in S(c) \cap B\left(\frac{rc}{2}\right)$ by Lemma 3.1. This shows that $\psi_n(t)$ is globally large for sufficiently large n. Due to the compactness of all minimizing sequence of (1.6), a contradiction to (3.16) is obtained. Theorem 1.2 has been proven.

4. Conclusion

In recent years, the nonlinear Schrödinger equation have been studied by many experts. This paper mainly adds a nonlinear term and a partial harmonic potential on this basis. In particular, the addition of nonlinear terms poses significant computational challenges, as the equations lose their compactness and translation invariance due to the presence of partial harmonic potentials. To solve the difficulty, we were inspired by the [8], the compactness of the minimization sequence is obtained by establishing the minimalization problem and using the concentration compactness principle, thus proving the stability of the standing waves for the equation in L^2-supercritical case.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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