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Boundedness of Sequentially Generated Measurable Products: A Neighborhood Approach

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Authors' contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

Many studies have been done on products of measurable sets. The most recent results highlight the properties of tensor products expressed as matrix products. Therefore, this study builds on existing research on product of measurable sets focusing on properties expressed in matrix products. This study investigates the conditions under which sequentially generated products of functions are measurably bound using $(\epsilon - \delta)$ criterion for uniform continuity . This article explores the connection between topological properties of measurable sets and boundedness of their products. The study sheds light on the application of r-neighborhood topological properties of refinement of measurable sets in determining the boundedness of sequentially generated products of measurable functions. Concepts such as monotonicity of functions, continuity from above of set functions, almost everywhere properties and r-neighborhood partition of measurable sets are applied in the context of p-integrable functions. The results of this research can be applied to develop the r-

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neighborhood business models where r represents the physical distance around a fixed business focal point that geographically creates a fruitful business environment for achievement of the optimal industrial and commercial profit margins determined by the boundedness of product functions. For a fixed product of functions i.e. the target of achievement, one can sequentially and by monotonicity of measurable functions determine the quantitative (or measurable) convergence of the product of functions which represents the interactive operational activities towards the defined business goals. Further, the results of this study can be applied in developing geometrical models in engineering by quantitative approximation to desired standards.

Keywords: Refinement, measurably bound; monotonically; decreasing; r-neighborhood.

1 Introduction

This study makes the sequence $(f_i \otimes \chi_A)_{i=1}^{\infty}$ of products of f_i and χ_A where $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ move quantitatively closer to $f \otimes x'$ for each i . With an appropriate choice of a real number $r > 0$, the r-neighborhood $N_r(x_i)$ of a point $x_i \in X$ as discussed in [1, 2] partitions an open set G_i for $i \in I$ [3, 4] such that the set

$$(L_p(\mu)^{f\otimes x'}_{f_j \otimes \chi_A})_{j=1}^{n} :\| (< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}\text{-}(< \mu_{|f|^p}(G_i), x' >)^{1/p}) \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

is made progressively smaller for large values of j.

Concepts on uniform continuity of functions (see[5]) and geometrical estimation of distance from a point to a given set (see[4, 6]) are utilized so that the quantity

$(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \parallel x' \parallel^{1/p'}$ is kept within the ϵ-distance of the quantity $(< \mu_{|f|^p}(G_i), x' >)^{1/p} \parallel x' \parallel^{1/p'}$ as we restrict a point $y \in G_i$ for each $i \in I$ to smaller intervals $N_r(x_i)$ of $x_i \in X$

While the continuity of functions is discussed, we study the behaviour of sequentially generated tensor products within the ϵ-distance parameter where the $(\epsilon - \delta)$ criterion for uniform continuity is applied.

2 Preliminaries

In this study, we consider $1 \leq p < \infty$ and the conjugate real number p' such that $1/p + 1p' = 1$ (see[3, 7, 8, 9]). Throughout this paper, (Ω, Σ, μ) denotes a measure space where Σ is a sigma ring of subsets of Ω, $\mu : \Sigma \to X$ is a countably additive vector measure, X a Banach space, $L_p(\mu)$ the space of p-integrable functions with respect to μ . The function $< x, x' >$ denotes the duality between the Banach space X with its topological dual X'. For each $x' \in X'$, we have $< \mu(A), X' >=< x, x' >$ for every $A \in \Sigma$ (see[3]). If a sequence $(f_n)_{i=1}^{\infty} \in L_p(\mu)$ and χ_A is the characteristic function of a measurable set A of finite measure, then $f_n \otimes \chi_A$ denotes the product of f_n and χ_A such that $f_n \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ for each n

Let $< \mu, x' >= \mu_{x'}$ for every $x' \in X'$ such that $\mu_{x'} = \mu_{x' \div \|x'\|}(\|x'\|)$

Considering the results in [[10], P. 10, Proposition 3.2] on integral representation of the vector measure duality function, we have

$(\int \mid f_j \mid^p .\chi_A \delta < \mu, x' >)^{1/p} \parallel x' \parallel^{1/p'} \parallel = (< \mu_{|f_j|^p}(A), x' \div \|x'\| >)^{1/p}(\parallel x' \parallel)$

$= (< \mu_{|f_j|^p}(A), x' >)^{1/p}(\parallel x' \parallel) \parallel x' \parallel^{-1/p}$

$= (< \mu_{|f_j|^p}(A), x' >)^{1/p} \parallel x' \parallel^{1/p'}$

Therefore, $(< \mu_{|f_j|^p}(A), x' >)^{1/p} \parallel x' \parallel^{1/p'}$ is well defined as demostrated in [3, 7, 4, 11, 12, 8, 13], where $\mu_{|f_j|^p}(A) \in X$ for every $A \in \Sigma$.

Definition 1(Refinement)(see [14])

A family $(A_j : j \in \alpha)$ of subsets of X is called a refinement of a set G if for every r_i-neighborhood $N_{r_i}(x_i)$ of a point x_i in A_i, there is a subset G_i in G such that

$(< \mu_{|f_j|^p}(N_{r_i}(x_i)), x' >)^{1/p} \leq < \mu_{|f_j|^p}(G_i), x' >)^{1/p}$ for each i and j

Therefore,

$(< \mu_{|f_j|^p}(\bigcup_{i=1}^{\infty} N_{r_i}(x_i)), x' >)^{1/p} = (< \mu_{|f_j|^p}(G_i), x' >)^{1/p}$

$\sum_{i=1}^{\infty}(< \mu_{|f_j|^p}(N_{r_i}(x_i)), x' >)^{1/p} = (< \mu_{|f_j|^p}(G_i), x' >)^{1/p}$

for each j

Definition 2(Measurably bound Products)

Let (Ω, Σ, μ) be a measure space and $E_{n_j+1} \subset E_{n_j}$ for each n_j be set of measurable sets each of which is a refinement of G such that $\mu(E_{n_1}) < \infty$ for all n_j. There exists a neighborhood $N_r(x_i)$ such that $N_r(x_i) \subset G_i \in G$.

Define $E_{n_j} = (L_p(\mu)^{f \otimes x'}_{f_j \otimes \chi_A})^n_{j=1} : \parallel [(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$- $(< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel \geq \epsilon)$

Such that

$(< \mu_{|f_j|^p}(E_n), x' >)^{1/p} = (< \mu_{|f_j|^p}(\bigcap_{j=1}^{\infty} E_{n_j}), x' >)^{1/p}$ where $E_n \downarrow \emptyset$ for each n.

Therefore,

$(< \mu_{|f_j|^p}(E_n), x' >)^{1/p} = (< \mu_{|f_j|^p}(\bigcap_{j=1}^{\infty} E_{n_j}), x' >)^{1/p}$

$\leq (< \mu_{|f_j|^p}(E_{n_j}), x' >)^{1/p} \leq (< \mu_{|f_j|^p}(E_{n_1}), x' >)^{1/p}$

for all n_j

As noted in [15, 4] regarding monotonically increasing sets , it follows that

$$(<\mu_{|f_j|^p}((E_{n_j})^c), x'>)^{1/p} \uparrow (<\mu_{|f_j|^p}((E_n)^c), x'>)^{1/p} \text{ for each } n$$

where $(E_{n_j})^c$ and $(E_n)^c$ are the complements of E_{n_j} and E_n respectively (for examples on complements of sets, see [1, 16]).

The results in [8, 15] on monotonically decreasing sets and on Banach space of locally integrable functions where $f_n \downarrow 0$ (see [18]) satisfy

$$(<\mu_{|f_j|^p}((E_{n_j})), x'>)^{1/p} \downarrow (<\mu_{|f_j|^p}((E_n)), x'>)^{1/p} \text{ for each } n_j$$

From the results in [17] on the convergence of measurable sets to zero with respect to $c*$-algebra valued measures, in (see [3]) on integral mappings and order continuous Banach spaces of integrable functions with respect to vecror measure and in [11, 9] on projective tensor products, the sequence $(f_i \otimes \chi_A)_{i=1}^{\infty}$ of products of measurable functions, where $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ is therefore said to be measurably bound to $f \otimes x'$ at a point x if

$$(L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| [(<\mu_{|f_j|^p}(N_r(x_i)), x'>)^{1/p} \text{-} (<\mu_{|f|^p}(G_i), x'>)^{1/p}] \| x' \|^{1/p'} \geq \epsilon)$$

is monotonically decreasing to a null set as $n \to \infty$ for $\epsilon > 0$

Definition 3(r-Neighborhood Partition)

Let $(G_i : i \in I)$ be a family of measurable open subsets of the normed space X and $(x_i)_{i=1}^{\infty}$ be a sequence of elements in G_i for each $i \in I$ where I is an index set. The sequence $(N_{r_1}(x_1), N_{r_2}(x_2), ..., N_{r_{i-1}}(x_{i-1}),$ $N_{r_i}(x_i), N_{r_{i+1}}(x_{i+1})...)$ of r_i - neighborhoods of x_i such that $N_{r_i}(x_i) \bigcap N_{r_j}(x_j) = \emptyset$ for $i \neq j$ is said to partition G_i into disjoint sets for each $i \in I$ if

$$N_{r_i}(x_i) \uparrow G_i \text{ for each } i \in I \text{ (see [15, 4])}.$$

Considering the collection $(N_{r_i}(x_i) : i = 1, 2, ...)$ of non-overlapping r_i - neighborhoods of x_i called partitions of G_i as illustrated in [6], then

$$\bigcup_{i=1}^{\infty} N_{r_i}(x_i) = G_i \text{ for each } i \in I.$$

If $r = \min\ (r_1, r_2, ...r_{i-1}, r_i, r_{i+1}, ...)$, then

$$N_r(x_i) \subset G_i \text{ for each } i \in I \text{ (see [1, 2])}$$

Definition 4(Directed Set of Vector Measure Duality)(see [11, 12])

A set $(<\mu_i, x'>)_{i=1}^{n}$ of non-negative vector measure duality is said to be increasingly directed if for $<\mu_i, x'>$ $\leq <\mu_{k_i}, x'>$ where $1 \leq i < k_i \leq n$ we have

$$<\mu_i(A), x'> = LUB_k <\mu_{k_i}(A), x'> \text{ for every } A \in \Sigma$$

Definition 4(Almost Everywhere Property)(see [3, 7, 8])

Let x be an element in X. A proposition $P(x)$ is true almost everywhere if there exists a null set E such that $x \in X \sim E$

3 Main Results

Proposition 1

Let $(f_n)_{i=1}^{\infty}$ and f be p-integrable functions such that f_n converges to f uniformly.The set $(f_i \otimes \chi_A)_{i=1}^{n}$ of measurable functions where $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ for each i is said to be measurably bound to $f \otimes x'$ at a point x if

$$(L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=m}^{n} :\| [(\textstyle\sum_{j=m}^{n} <\mu_{|f_j|^p}(N_r(x_i)), x'>)^{1/p} \text{ - } (<\mu_{|f|^p}(G_i), x'>)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon/2)$$

is monotonically decreasing to a null set as $n \to \infty$ for $\epsilon > 0$

Proof

Since f_n converges to f uniformly and $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ for $i = 1, ..., n$ is measurably bound to $f \otimes x'$ at a point x (by hypothesis), we need to sequentially show that set $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ and $f \otimes x'$ which are at a distance greater than an arbitrary real number $\epsilon > 0$, can be made progressively smaller for values that are in some r-neighborhood $N_r(x_i)$ of $x_i \in X$ (see [1, 2]). Further, the $(\epsilon - \delta)$ criterion on uniform continuity of functions (see [5]) is apllied to obtain the desired results. Therefore, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\nabla^{f_n} = (L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| [(\textstyle\sum_{j=1}^{n} <\mu_{|f_j|^p}(N_r(x_i)), x'>)^{1/p} \text{ - } (<\mu_{|f|^p}(G_i), x'>)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

is monotonically decreasing to a null set as $n \to \infty$

provided $y \in N_\delta(x_i) \; \forall \; y \in G_i$.

Suppose we choose $n > m$ such that

$$\nabla^{f_{m-1}} = (L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{m-1} :\| [(\textstyle\sum_{j=1}^{m-1} <\mu_{|f_j|^p}(N_r(x_i)), x'>)^{1/p}\text{- } (<\mu_{|f|^p}(G_i), x'>)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon/2)$$

The choice of $\epsilon > 0$ and $\epsilon/2 > 0$ Implies that

$\nabla^{f_{m-1}} \subseteq \nabla^{f_n}$

Therefore,

$(L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\sum_{j=1}^n < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$ - $(\sum_{j=1}^m < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon/2)$

Fix n, taking limits as $m \to \infty$ and applying Cauchy criterion as discussed

in [3, **?**], we obtain

$$(L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=m} :\| [(\sum_{j=m}^n < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} - (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon/2)$$

$$= \nabla^{f_{m-1}} \bigcap \nabla^{f_n}$$

whenever $y \in N_\delta(x_i)$ $\forall$ $y \in G_i$

Therefore,

$$(L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=m} :\| (\sum_{j=m}^n < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \parallel x' \parallel^{1/p'}$$

$$- (< \mu_{|f|^p}(G_i), x' >)^{1/p} \parallel x' \parallel^{1/p'} \| \geq \epsilon/2) \subseteq \nabla^{f_n} \downarrow \emptyset$$

as $n \to \infty$

Hence,

$$(L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=m} :\| [(\sum_{j=m}^n < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \parallel x' \parallel^{1/p'}$$

$$- (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon/2) \downarrow \emptyset$$

as $n \to \infty$ provided $y \in N_\delta(x_i)$ $\forall$ $y \in G_i$

Proposition 2

Let $(f_n)_{i=1}^\infty$, f and g be p-integrable functions such that the set $(f_1 \otimes \chi_A ... f_n \otimes \chi_A)$ of products where $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ for each i is measurably bound to $f \otimes x'$ and $g \otimes x'$ at a point x. If for $\epsilon > 0$, we have

$\parallel (< \mu_{|f|^p}(G_i), x' >)^{1/p}$ - $(< \mu_{|g|^p}(G_i), x' >)^{1/p} \parallel < \epsilon$

where $(G_i : i \in I)$ is a family of measurable open subsets of the normed space X, then

$(< \mu_{|f|^p}(G_i), x' >)^{1/p} = (< \mu_{|g|^p}(G_i), x' >)^{1/p}$ a.e.

Proof

Let $(\nabla)^g_f = ((L_p(\mu)^{g\otimes x'}_{f\otimes x'}) :\parallel [(< \mu_{|f|^p}(G_i), x' >)^{1/p}$ - $(< \mu_{|g|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel < \epsilon)$
for each $i \in I$

Let the collection $(A_j : j \in \alpha)$ be a refinement of a set G. For every r_i-neighborhood $N_{r_i}(x_i)$ of a point x_i in A_j, there exists a subset G_i in G such that $N_{r_i}(x_i \subseteq G_i$ for each i and j (see [14])

The δ-criterion for generation of a closed unit ball and a unit sphere in a Banach space as demostrated in [19] is applied constructing r_i-neighborhood of a point x_i as follows

Take $r = 1/2$ min $(r_1, r_2, ..., r_n)$. It follows that $r > 0$ and

$N_r(x_i) \subseteq N_{r_i}(x_i) \subseteq G_i$ for each $i \in I$

On application of $(\epsilon - \delta)$ criterion on uniform continuity as discussed in [20] and the duality function $< \mu, x' >$ (see [3, 7, 12, 8, 13]), we take y closer to x_i such that for $\delta > 0$, we have

$y \in N_\delta(x_i) \ \forall \ y \in G_i$

Consequently,

$(L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\parallel [(\sum^n_{j=1} < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p})$ - $(< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel \geq \epsilon/2)$

Let $\nabla^f_{f_n} = (L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\parallel [(\sum^n_{j=1} < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$ - $(< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel \geq \epsilon/2)$

and

$\nabla^g_{f_n} = (L_p(\mu)^{g\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\parallel [(\sum^n_{j=1} < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$ - $(< \mu_{|g|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel \geq \epsilon/2)$

Since $\parallel [(< \mu_{|f|^p}(G_i), x' >)^{1/p}$

- $(< \mu_{|g|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \parallel < \epsilon$ on $(\nabla)^g_f$

for each $i \in I$, it follows that

$$((\nabla)_f^g)^c = ((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) :\| [(<\mu_{|f|^p}(G_i), x'>)^{1/p} - (<\mu_{|g|^p}(G_i), x'>)^{1/p}] \| x' \|^{1/p'} \| \geq \epsilon)$$

Therefore,

$$((\nabla)_f^g)^c \subseteq \nabla_{f_n}^f \bigcup \nabla_{g_n}^f$$

Since the set

$(f_1 \otimes \chi_A ... f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ and $g \otimes x'$ at a point x (by hypothesis), it follows that

$\nabla_{f_n}^f$ and $\nabla_{g_n}^f$ are both monotonically $\downarrow \emptyset$ as $n \to \infty$

Subsequently,

The $<\mu, x'>$ - measure of $((\nabla)_f^g)^c$ is zero,

provided $y \in N_\delta(x_i)\ \forall\ y \in G_i$

Suppose,

$$((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) :\| [(<\mu_{|f|^p}(G_i), x'>)^{1/p} - (<\mu_{|g|^p}(G_i), x'>)^{1/p}] \| x' \|^{1/p'} \| \neq 0)$$

$$= \bigcup_{k=1}^{\infty} ((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) :\| (<\mu_{|f|^p}(G_i), x'>$$

$$- (<\mu_{|g|^p}(G_i), x'>)^{1/p}) \| x' \|^{1/p'} \| \geq 1/k) = 0$$

It follows that

$$((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) :\| [(<\mu_{|f|^p}(G_i), x'>)^{1/p} - (<\mu_{|g|^p}(G_i), x'>)^{1/p}] \| x' \|^{1/p'} \| \neq 0)$$

is a null set since it is equal to the countable union of null sets (see [3, 4]).

The results on almost everywhere property in [3, 7, 8] demostrate that

$$(<\mu_{|f|^p}(G_i), x'>)^{1/p} = (<\mu_{|g|^p}(G_i), x'>^{1/p} \text{ a.e.}$$

Proposition 3

Let $(f_n)_{i=1}^{\infty}$, f and g be p-integrable functions such that the set $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ of products where $f_i \otimes \chi_A \in L_p(\mu) \otimes \Sigma$ for $i = 1, ..., n$ is measurably bound to $f \otimes x'$ at a point x. If

$$((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) : (< \mu_{|f|^p}(G_i), x' >)^{1/p} \neq (< \mu_{|g|^p}(G_i), x' >)^{1/p})$$

where G_i for each $i \in I$ is null set, then $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $g \otimes x'$ at a point x

Proof

Let E denote the set

$$((L_p(\mu)_{f\otimes x'}^{g\otimes x'}) : (< \mu_{|f|^p}(G_i), x' >)^{1/p} \neq (< \mu_{|g|^p}(G_i), x' >)^{1/p}$$

Then E is a null set (by hypothesis). From the results on almost everywhere

property in [3, 7, 8], it follows that

$$< \mu_{|f|^p}(G_i), x' >)^{1/p} = (< \mu_{|g|^p}(G_i), x' >)^{1/p} \text{ a.e.}$$

Let $F_n = ((L_p(\mu)_{f_j\otimes\chi_A}^{g\otimes x'})_{j=1}^n :\| (\sum_{j=1}^n [< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$ - $(< \mu_{|g|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$

$\subseteq E \bigcup (L_p(\mu)_{f_j\otimes\chi_A}^{f\otimes x'})_{j=1}^n :\| (\sum_{j=1}^n < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \parallel x' \parallel^{1/p'}$

- $(< \mu_{|f|^p}(G_i), x' >)^{1/p} \parallel x' \parallel^{1/p'} \| \geq \epsilon)$

Since E is null set and $(f_1 \otimes \chi, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ at a point x by hypothesis), then

$((L_p(\mu)_{f_j\otimes\chi_A}^{g\otimes x'})_{j=1}^n :\| ((\sum_{j=1}^n [(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$ - $(< \mu_{|g|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$

From which the results follows (see Proposition 1) that

$$F_n \downarrow \emptyset \text{ as } n \to \infty$$

Proposition 4

Let $(f_n)_{i=1}^{\infty}$, $(g_n)_{i=1}^{\infty}$ and f be p-integrable functions such that the set $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ at a point x. If

$$((L_p(\mu)_{f_j \otimes \chi_A}^{g_i \otimes \chi_A}) :\| [(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|g_i|^p}(N_r(x_i)), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| = 0)$$

is a non empty set for each $j = 1, ..., n$, then given a real $\epsilon > 0$, the set

$$((L_p(\mu)_{g_j \otimes \chi_A}^{f \otimes x'}) :\| [(\sum_{j=1}^{n} < \mu_{|g_j|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

Proof

The result in (see [21]) on linearly independent sequences of the form $(f_n)_{i=1}^{\infty}$ and $(g_n)_{i=1}^{\infty}$ in a Hilbert space $H_1 \otimes H_2$ and the associated representation in terms of bounded operators is applied in proving this proposition.

Let $E_n = ((L_p(\mu)_{f_j \otimes \chi_A}^{g_i \otimes \chi_A}) :\| [(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|g_i|^p}(N_r(x_i)), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| = 0)$

Since $E_n \neq \emptyset$ (by hypothesis), then by almost everywhere property for pairwise distinct sets as discussed in [3, 7, 8], it follows that

$$(E_n)^c = ((L_p(\mu)_{f_j \otimes \chi_A}^{g_i \otimes \chi_A}) : (< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p}$$

$$\neq (< \mu_{|g_i|^p}(N_r(x_i)), x' >)^{1/p}$$

is a null set.

Suppose

$$((L_p(\mu)_{g_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| [(\sum_{j=1}^{n} < \mu_{|g_j|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

$$\subseteq (E_n)^c \bigcup ((L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| (\sum_{j=1}^{n} < \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \parallel x' \parallel^{1/p'}$$

$$\text{- } (< \mu_{|f|^p}(G_i), x' >)^{1/p} \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

It can be deduced from the results in [18] and [10], p. 15] on integrable functions which are equal almost everywhere that $(E_n)^c$ is a null set.

Therefore, $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ at x. It follows from Proposition 3 that

$$((L_p(\mu)_{g_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| [(\sum_{j=1}^{n} < \mu_{|g_j|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

is monotonically $\downarrow \emptyset$ as $n \to \infty$ as required

Proposition 5

Let the family $(A_j : j \in \alpha)$ of subsets of X be a refinement of a set G_i for each i and $(f_n)_{i=1}^{\infty}$, f be p-integrable functions such that the set $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ at $x \in X$. If for given real numbers $\beta > 0$ and $\epsilon > 0$, the set

$$((L_p(\mu)^{f\otimes x'}_{f_k\otimes\chi_A})_{k\geq n} :\| [(\sum_{k\geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \|\geq 1/\beta)$$

then for $\epsilon(\beta) > 0$,

$$\| [(\sum_{k\geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \|< \epsilon$$

Proof

Since $(A_j : j \in \alpha)$ is a refinement of a set G_i for each i (by hypothesis), then for every r-neighborhood $N_r(x_i)$ of a point x_i in A_i, there is a subset G_i in G such that $N_r(x_i) \subseteq G_i$ for each i and j (see [14])

For each natural number i, each $N_r(x_i)$ has non empty lower bound B in G_i such that

$$(< \mu_{|f_j|^p}(B), x' >)^{1/p} \| x' \|^{1/p'} < \epsilon$$

$$(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \| x' \|^{1/p'} < \epsilon$$

Therefore,

$$lim_{j\to\infty}(< \mu_{|f_j|^p}(N_r(x_i)), x' >)^{1/p} \| x' \|^{1/p'} < \epsilon$$

Define $E^{\beta}_n = \bigcup_k((L_p(\mu)^{f\otimes x'}_{f_k\otimes\chi_A})_{k\geq n} :\| [(\sum_{k\geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p}$

$$\text{- } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \|\geq 1/\beta)$$

The above inequality satisfy the j-convergence properties [see [22] of measurable products , where the norm difference between sequentially generated measurable products monotonically becomes zero.

Since $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ at $x \in X$, as noted in proposition 1, there exists a real number $\beta > 0$ satisfying

$$((L_p(\mu)^{f\otimes x'}_{f_k\otimes\chi_A})_{k\geq n} :\| [(\sum_{k\geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p} \text{ - } (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \|\geq 1/\beta) \downarrow \emptyset \text{ as } n \to \infty$$

Subsequently,

$$(< \mu_{|f|^p}(E_n^\beta), x' >)^{1/p} \downarrow 0$$

Define $E = \bigcup_{\beta=1}^{\infty} E_n^\beta$

$E^c = \bigcap_{\beta=1}^{\infty}(E_n^\beta)^c$ where E^c is the complement of E in $L_p(\mu) \otimes X'$

Applying the results on topological properties of integrable functions with respect to a Banach space valued measure which have been extensively studied in [20], it follows that

$$E^c = \bigcap_{\beta=1}^{\infty} \bigcap_k ((L_p(\mu)_{f_k \otimes \chi_A}^{f \otimes x'})_{k \geq n} :\| [(\textstyle\sum_{k \geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p}$$

$$- (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \| \geq 1/\beta)^c$$

$$= \bigcap_{\beta=1}^{\infty} \bigcap_k ((L_p(\mu)_{f_k \otimes \chi_A}^{f \otimes x'})_{k \geq n} :\| [(\textstyle\sum_{k \geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p}$$

$$- (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \| < 1/\beta)$$

Let $1/\beta < \epsilon$ for $\beta > 0$

Therefore,

$$\bigcap_{\beta=1}^{\infty} \bigcap_k ((L_p(\mu)_{f_k \otimes \chi_A}^{f \otimes x'})_{k \geq n} :\| [(\textstyle\sum_{k \geq n} < \mu_{|f_k|^p}(N_r(x_i)), x' >)^{1/p} - (< \mu_{|f|^p}(G_i), x' >)^{1/p}] \| x' \|^{1/p'} \| < \epsilon)$$

On E^c.

Corollary 1

Let $(f_n)_{i=1}^{\infty}$, $(g_n)_{i=1}^{\infty}$ and f be p-integrable functions with respect to an increasingly directed scalar measure $< \mu_i, x' >$ for each i and $(G_i : i \in I)$ be a family of closed measurable subsets of X. If $(f_1 \otimes \chi_A, ..., f_n \otimes \chi_A)$ is measurably bound to $f \otimes x'$ over the r-neighborhood $N_r(x_i)$ of a point x_i in $A \in \Sigma$, then

$$((L_p(\mu)_{f_j \otimes \chi_A}^{f \otimes x'})_{j=1}^{n} :\| [(\textstyle\sum_{j=1}^{n} < \mu_{k_j} | f_j |^p (N_r(x_i)), x' >)^{1/p}$$

$$- (< \mu_{k_j} | f |^p (X \sim G_i), x' >)^{1/p}] \| x' \|^{1/p'} \| \geq \epsilon) \downarrow \emptyset$$

Proof

Since G_i is closed for each i, then $G^C = X \sim G_i$ is topologically open follows from the results in [1, 16, 2].

By applying the property of non-overlapping open sets and the intersection of their interiors as noted in [6], p. 2], we have

$$A \bigcap G_i^c = A \sim G_i \neq \emptyset$$

If $r = \min\ (r_1, r_2, ...r_n)$, the results in [1, 16, 2] demonstrate that for a topologically open set G_i, there exists an r-neighborhood $N_r(x_i)$ such that

$N_r(x_i) \subset G_i$ for $r > 0$ and for each i

By definition 2 on measurably bound products and using the results in [23], Definition 1.3] on geometrically doubling metric spaces, we conclude that for any set G_i in X, there is r-neighborhood $N_r(x_i)$ covering G_i such that

$$((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\textstyle\sum_{j=1}^n < \mu_j| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-}\ (< \mu_j| \ f \ |^p(G_i^c), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon) \downarrow \emptyset$$

The results discussed in [11, 12] on increasingly directed set of vector measure duality and the supremum property taken over the r-neighborhoods covering G_i [24] ,p. 6] are applied in constructing the following inclusion

$$((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\textstyle\sum_{j=1}^n < \mu_j| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-}\ (< \mu_j| \ f \ |^p(G_i^c), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

$$= LUB_k((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\textstyle\sum_{j=1}^n < \mu_{k_j}| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-}\ (< \mu_{k_j}| \ f \ |^p(G_i^c), x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

$$= LUB_k((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| (\textstyle\sum_{j=1}^n < \mu_{k_j}| \ f_j \ |^p(N_r(x_i)), x >^{1/p}$$

$$\text{-}\ (< \mu_{k_j}| \ f \ |^p[(A \bigcup G_i)^c) \bigcup (X \sim G_i)], x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

$$\subseteq LUB_k((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\textstyle\sum_{j=1}^n < \mu_{k_j}| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-}\ (< \mu_{k_j}| \ f \ |^p[(A \bigcup G_i)^c)], x' >)^{1/p}$$

$$+ (< \mu_{k_j}| \ f \ |^p[(X \sim G_i)], x' >)^{1/p}] \parallel x' \parallel^{1/p'} \| \geq \epsilon)$$

Therefore

$$((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\sum_{j=1}^n < \mu_{k_j}| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-} \ (< \mu_{k_j}| \ f \ |^p[(X \sim G_i)], x' >)^{1/p}] \ \| \ x' \ \|^{1/p'} \| \geq \epsilon)$$

$$\subseteq ((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| \ (\sum_{j=1}^n < \mu_j| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-} \ (< \mu_j| \ f \ |^p(G_i^c), x' >)^{1/p}] \ \| \ x' \ \|^{1/p'} \| \geq \epsilon)$$

The preceding discussion satisfy the monotone property in [15, 16] from which the results follows

$$((L_p(\mu)^{f\otimes x'}_{f_j\otimes\chi_A})^n_{j=1} :\| [(\sum_{j=1}^n < \mu_{k_j}| \ f_j \ |^p(N_r(x_i)), x' >)^{1/p}$$

$$\text{-} \ (< \mu_{k_j}| \ f \ |^p(X \sim G_i), x' >)^{1/p}] \ \| \ x' \ \|^{1/p'} \| \geq \epsilon) \downarrow \emptyset$$

4 Conclusion

The results obtained in this study highlight the boundedness of sequentially generated measurable products at a point using the $\epsilon - \delta$ criterion. The article explored the connection between topological properties of measurable sets, measurable functions and boundedness of products of measurable functions. The research is unique in the sense that it considers the convergence of sequentially generated measurable products in the context of monotonically decreasing sets. This research can be extended to sections of divergence of n-dimensional products.

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Competing Interests

Author has declared that no competing interests exist.

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