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The Study of Mean Value Theorems of K-analytic Functions

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we obtain differential and integral mean value theorems for K-analytic functions, generalizing the corresponding results on analytic and conjugate analytic functions.

Keywords: K-analytic functions; differential mean value theorem; integral mean value theorem.

MSC (2010): 26A24, 26B35.

1 Introduction

Based on analytic functions, Chen [1] introduced conjugate analytic functions, Zhang [2] extended the definitions of analytic functions and conjugate analytic functions to K-analytic functions. For more knowledge of conjugate

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analytic functions and K-analytic functions, one can see [3, 4, 5, 6]. Differential and integral mean value theorems are important tools in analysis. Differential and integral mean value theorems as ones in real analysis do not hold in complex functions. For example, let $f(z) = e^{iz}$. Then $f(2\pi) - f(0) = 0$, however $f'(z) = ie^{iz} \neq 0$. This example can show the integral mean value theorem false in complex functions. Thus, the study of differential and integral mean value theorems in the theory of complex functions lead to the interests of mathematicians. Differential and integral mean value theorems in analytic and conjugate analytic functions have been studied in [7]. We are dedicated to getting differential and integral mean value theorems for K-analytic functions to extend the corresponding results in [7]. The following definitions were introduced in [2] and [8].

Definition 1.1. The forms of complex number as $x + iky$ ($k \in \mathbb{R}$, $k \neq 0$) are called K-complex number of $x + iy$, denoted by $z(k)$.

Definition 1.2. If $\lim_{\Delta z(k)\to 0} f(z) = f(z_0)$, then we call that $f(z)$ is K-continuous at z_0.

Definition 1.3. Let the function $f(z)$ be defined in a neighborhood of z_0. If

$$\lim_{\Delta z(k)\to 0} \frac{\Delta f}{\Delta z(k)} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z(k) - z_0(k)}$$

exists, then we call that $f(z)$ is K-differential at z_0, the limit is the K-derivative of $f(z)$ at z_0, denoted by $f'_{(k)}(z_0)$ or $\frac{df(z)}{z(k)}|_{z=z_0}$, i.e.,

$$f'_{(k)}(z_0) = \frac{df(z)}{z(k)}\Big|_{z=z_0} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z(k) - z_0(k)}.$$

Definition 1.4. If $f(z)$ is K-differential in a domain D, we say that $f(z)$ is analytic in D; If $f(z)$ is K-analytic in a neighborhood of z_0, then we say that $f(z)$ is analytic at z_0.

Definition 1.5. Let a curve $C(\alpha, \beta)$: $z = z(t)$ ($a \leq z \leq b$) be oriented with the starting point $\alpha = z(a)$ and ending point $\beta = z(b)$. Choose $n - 1$ points $a = z_0$, $z_1, z_2, ..., z_{n-1}, z_n = b$ sequently in the curve $C(a, b)$ separating C to n arcs $\widehat{z_{i-1}z_i}$, $1 \leq i \leq n$. For any $\xi_i \in \widehat{z_{i-1}z_i}$ ($1 \leq i \leq n$), give a sum

$$\sum_{i=1}^{\infty} f(\xi_i)\Delta_i z(k),$$

where $\Delta_i z(k) = z_i(k) - z_{i-1}(k)$. Let $d = \max_{1\leq i\leq n} |\overline{z_{i-1}z_i}|$. If

$$\lim_{d\to 0} \sum_{i=1}^{\infty} f(\xi_i)\Delta_i z(k)$$

exists, then we say that $f(z)$ is K-integrable on $C(a, b)$, and this value is called K-integral of f on $C(a, b)$, denoted by

$$\int_C f(z)dz(k).$$

2 Main Results

In this section, we give differential and integral mean value theorems of K-analytic functions on line segments and smooth curves, which generalize the corresponding results of [7, Section 2 and Section 3].

Theorem 2.1. *Let $f(z)$ be a K-analytic function in a domain D, a line segment $C[\alpha, \beta] \subset D$. Then there exist $\xi, \eta \in [\alpha, \beta]$ such that*

$$\int_C f(z)dz(k) = [\mathrm{Re}(\beta - \alpha) + ki\mathrm{Im}(\beta - \alpha)](\mathrm{Re}f(\xi) + i\mathrm{Im}f(\eta)).$$

Proof. Since $C[\alpha,\beta]$ is a line segment, it can be expressed by a parametric equation:

$$z = z(t) = \alpha + t(\beta - \alpha), \quad 0 \le t \le 1.$$

According to the method of calculating integral for a parametric equation, we have

$$\int_C f(z)dz(k) = \int_0^1 f(z(t))[(z'(t))(k)]dt.$$

But

$$(z'(t))(k) = \mathrm{Re}(\beta - \alpha) + ki\mathrm{Im}(\beta - \alpha),$$

thus

$$\int_C f(z)dz(k) = [\mathrm{Re}(\beta - \alpha) + ki\mathrm{Im}(\beta - \alpha)] \int_0^1 \{\mathrm{Re}f(\alpha + t(\beta - \alpha)) + i\mathrm{Im}f(\alpha + t(\beta - \alpha))\}dt.$$

For $f(z)$ is K-analytic which implies that $f(z)$ is K-continuous, by the first integral mean value theorem in real analysis, there exist two number $t_1,\ t_2 \in [0,1]$ such that

$$\int_C f(z)dz(k) = [\mathrm{Re}(\beta - \alpha) + ki\mathrm{Im}(\beta - \alpha)]\{\mathrm{Re}f(\alpha + t_1(\beta - \alpha)) + i\mathrm{Im}f(\alpha + t_2(\beta - \alpha))\}.$$

Let $\alpha + t_1(\beta - \alpha) = \xi$, $\alpha + t_2(\beta - \alpha) = \eta$. Thus, we complete the proof. □

Remark 2.1. Let $f(z) = e^{iz(k)}$, $D = \mathbf{C}$. Then f is a K-analytic function in D and thus satisfies the condition in Theorem 2.1.

Lemma 2.1. *Let $a, b, h_1 \in C$, $h_2 \in \mathbb{R}$. Then*

$$(a + b)(k) = a(k) + b(k), \quad (h_2h_1)(k) = h_2(h_1(k)).$$

Proof. Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $h_1 = h_{11} + ih_{12}$. According to Definition 1.1, we have

$$(a + b)(k) = a_1 + b_1 + ik(a_2 + b_2) = a_1 + ika_2 + b_1 + ikb_2 = a(k) + b(k).$$

$$h_2h_1(k) = (h_2h_{11} + ih_2h_{12})(k) = h_2h_{11} + ikh_2h_{12} = h_2(h_1(k)).$$

□

Theorem 2.2. *Let a K-analytic function $f(z)$ be a univalent function in D, line segment $C[\alpha,\beta] \subset D$ with the representation:*

$$z = z(t) = \alpha + t(\beta - \alpha), \quad 0 \le t \le 1, \tag{2.1}$$

the function $f(z)$ map $C[\alpha,\beta]$ to a curve γ, and a function $g(z)$ be continuous on γ. Then

(1) when $k = -1$, there exist $\xi, \eta \in [\alpha,\beta]$, such that

$$\int_\gamma g(u)du(k) = (\beta - \alpha)\left\{\mathrm{Re}(g(f(\xi)))\overline{f'_{(-1)}(\xi)} + i\mathrm{Im}(g(f(\eta)))\overline{f'_{(-1)}(\eta)}\right\}.$$

(2) when $k \ne -1$, there exist $\xi_i \in [\alpha,\beta]$, $1 \le i \le 4$, such that

$$\int_\gamma g(u)du(k) = (\mathrm{Re}(\beta - \alpha) + ik^2\mathrm{Im}(\beta - \alpha))[\rho_1(\xi_1)h_1(\xi_1) + i\rho_1(\xi_2)h_2(\xi_2)]$$

$$+ik(\beta - \alpha)[\rho_1(\xi_3)h_2(\xi_3) + i\rho_2(\xi_4)h_1(\xi_4)],$$

where

$$\rho_1(\xi) = \mathrm{Re}(g(f(\xi))), \quad \rho_2(\xi) = \mathrm{Im}(g(f(\xi))),$$
$$h_1(\xi) = \mathrm{Re}(f'_{(k)}(\xi)), \quad h_2(\xi) = \mathrm{Im}(f'_{(k)}(\xi)).$$

Proof. From the assumption on γ, the curve γ can be parameterized:

$$\gamma(t) = f(z(t)), \quad 0 \leq t \leq 1.$$

In the light of $f(z)$ being analytic on D, thus by the definition of K-derivative,

$$\begin{aligned}\frac{d\gamma(t)}{dt} &= \lim_{\Delta t \to 0} \frac{f(z(t+\Delta t)) - f(z(t))}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(z(t+\Delta t)) - f(z(t))}{(z(t+\Delta t) - z(t))(k)} \cdot \frac{(z(t+\Delta t) - z(t))(k)}{\Delta t} \\ &= f'_{(k)}(z(t))(z'(t))(k).\end{aligned} \tag{2.2}$$

Noting that $f(z)$ is a univalent K-analytic and the representation of the line segment C is (2.1), we know by (2.2) that Γ is a smooth curve, i.e., $\gamma'(t)$ is a continuous function with respect to t and $\gamma'(t) \neq 0$. Hence,

$$\begin{aligned}\int_\gamma g(u)du(k) &= \int_0^1 g(f(z(t))) \left(\frac{df(z(t))}{dt}\right)(k)dt \\ &= \int_0^1 g(f(z(t)))[f'_{(k)}(z(t))z'(t)(k)](k)dt.\end{aligned}$$

(1) If $k = -1$, then

$$[f'_{(k)}(z(t))z'(t)(k)](k) = [f'_{(-1)}(z(t))z'(t)(-1)](-1) = \overline{f'_{(-1)}(z(t))}z'(t).$$

Thus

$$\begin{aligned}\int_\gamma g(u)du(k) &= \int_0^1 g(f(z(t)))\overline{f'_{(-1)}(z(t))}z'(t)dt \\ &= (\beta - \alpha)\int_0^1 g(f(z(t)))\overline{f'_{(-1)}(z(t))}dt \\ &= (\beta - \alpha)\int_0^1 \left[\mathrm{Re}\left(g(f(z(t)))\overline{f'_{(-1)}(z(t))}\right) + i\mathrm{Im}\left(g(f(z(t)))\overline{f'_{(-1)}(z(t))}\right)\right] dt.\end{aligned}$$

From the assumptions, we know that $g(f(z(t)))\overline{f'_{(-1)}}(z(t))$ is continuous on $t \in [0, 1]$, therefore the real and imaginary part of it are continuous functions. Thus using the integral mean value on the real and imaginary parts of it respectively shows that there exist two numbers $t_1,\ t_2$ such that

$$\begin{aligned}&\int_0^1 \left[\mathrm{Re}\left(g(f(z(t)))\overline{f'_{(-1)}(z(t))}\right) + i\mathrm{Im}\left(g(f(z(t)))\overline{f'_{(-1)}(z(t))}\right)\right] dt \\ = \quad &\mathrm{Re}\left(g(f(z(t_1)))\overline{f'_{(-1)}(z(t_1))}\right) + i\mathrm{Im}\left(g(f(z(t_2)))\overline{f'_{(-1)}(z(t_2))}\right).\end{aligned}$$

Let $\xi = z(t_1)$, $\eta = z(t_2)$. Then

$$\int_\gamma g(u)du(k) = (\beta - \alpha)\left[\mathrm{Re}\left(g(f(\xi))\overline{f'_{(-1)}(\xi)}\right) + i\mathrm{Im}\left(g(f(\eta))\overline{f'_{(-1)}(\eta)}\right)\right].$$

(2) If $k \neq -1$, we use Lemma 2.1 to have

$$\begin{aligned}&\int_\gamma g(u)du(k) = \int_0^1 g(f(z(t)))\left(\frac{df(z(t))}{dt}\right)(k)dt \\ = \quad &\int_0^1 \Big\{\Big(\rho_1(z(t)) + i\rho_2(z(t))\Big)h_1(z(t))[\mathrm{Re}(\beta - \alpha) + ik^2\mathrm{Im}(\beta - \alpha)] \\ &+\Big(\rho_1(z(t)) + i\rho_2(z(t)))h_2(z(t)\Big)[-k\mathrm{Im}(\beta - \alpha) + ik\mathrm{Re}(\beta - \alpha)]\Big\}dt \\ = \quad &[\mathrm{Re}(\beta - \alpha) + ik^2\mathrm{Im}(\beta - \alpha)]\int_0^1 \Big(\rho_1(z(t)) + i\rho_2(z(t))\Big)h_1(z(t))dt \\ &+ki(\beta - \alpha)\int_0^1 [\rho_1(z(t)) + i\rho_2(z(t))]h_2(z(t))dt.\end{aligned}$$

Since the functions $\rho_1(z(t)),\ \rho_2(z(t)),\ h_1(z(t)),\ h_2(z(t))$ are real-valued continuous functions, by integral mean value theorem of real functions, there exist $t_i,\ 3 \le i \le 6$, such that

$$\int_0^1 \left(\rho_1(z(t)) + i\rho_2(z(t))\right) h_1(z(t))dt = \rho_1(z(t_3))h_1(z(t_3)) + i\rho_2(z(t_4))h_1(z(t_4)),$$

$$\int_0^1 [\rho_1(z(t)) + i\rho_2(z(t))]h_2(z(t))dt = \rho_1(z(t_5))h_2(z(t_5)) + \rho_2(z(t_6))h_2(z(t_6)).$$

Thus,

$$\int_\gamma g(u)du(k) = [\mathrm{Re}(\beta-\alpha) + ik^2\mathrm{Im}(\beta-\alpha)]\left(\rho_1(z(t_3))h_1(z(t_3)) + i\rho_2(z(t_4))h_1(z(t_4))\right)$$
$$+ki(\beta-\alpha)\Big(\rho_1(z(t_5))h_2(z(t_5)) + i\rho_2(z(t_6))h_2(z(t_6))\Big).$$

Let $z(t_3) = \xi_1,\ z(t_4) = \xi_2,\ z(t_5) = \xi_3,\ z(t_6) = \xi_4$. Then

$$\begin{aligned}\int_\gamma g(u)du(k) \quad &= \quad [\mathrm{Re}(\beta-\alpha) + ik^2\mathrm{Im}(\beta-\alpha)]\left(\rho_1(\xi_1)h_1(\xi_1)) + i\rho_2(\xi_2)h_1(\xi_2)\right) \\ &\quad\quad +ki(\beta-\alpha)\Big(\rho_1(\xi_3)h_2(\xi_3) + \rho_2(\xi_4)h_2(\xi_4)\Big).\end{aligned}$$

The proof is finished. □

Remark 2.2. When $k = -1$, the K-analytic function is just the conjugate analytic function. [7, Theorem 3.3.3] got the integral mean value on a smooth curve of the conjugate function. But we should indicate that the result has an error, i.e., the coefficient $\beta - \alpha$ should be $\overline{\beta - \alpha}$.

Remark 2.3. Let $f(z) = e^{iz(k)}$, $D = \{z \in \mathbf{C} : 0 < \mathrm{Re}z < 2\pi, -\infty < \mathrm{Im}z < \infty\}$. Then f is a univalent K-analytic function in D and thus satisfies the conditions in Theorem 2.2.

Theorem 2.3. *Let a function $f(z)$ is a K-analytic function on domain D, line segment $C(\alpha,\beta) \subset D$. Then there exist $\xi,\ \eta \in C[\alpha,\beta]$, such that*

$$\frac{f(\beta) - f(\alpha)}{\mathrm{Re}(\beta-\alpha) + ik\mathrm{Im}(\beta-\alpha)} = \mathrm{Re}f'_{(k)}(\xi) + i\mathrm{Im}f'_{(k)}(\eta).$$

Proof. Let the parametric equation of the line segment $C(\alpha,\beta)$ be:

$$z = z(t) = \alpha + t(\beta-\alpha), \quad 0 \le t \le 1.$$

Suppose that

$$F(t) = \frac{f(\alpha + t(\beta-\alpha))}{\mathrm{Re}(\beta-\alpha) + ik\mathrm{Im}(\beta-\alpha)} = U(t) + iV(t),\ 0 \le t \le 1.$$

Due to $f(z)$ being K-analytic on D, (2.2) and the fact

$$z'(t)(k) = \mathrm{Re}(\beta-\alpha) + ik\mathrm{Im}(\beta-\alpha),$$

it follows that

$$\begin{aligned}\frac{dF(t)}{dt} \quad &= \quad \frac{f'_{(k)}(z(t))z'(t)(k)}{\mathrm{Re}(\beta-\alpha) + ik\mathrm{Im}(\beta-\alpha)} \\ &= \quad f'_{(k)}(z(t)) \\ &= \quad U'(t) + iV'(t).\end{aligned}$$

So $U(t)$ and $V(t)$ satisfy the conditions of Lagrange mean value theorem in real analysis and thus there exist two numbers $t_1,\ t_2 \in \mathbb{R}$ such that

$$\begin{aligned} & \frac{f(\beta)-f(\alpha)}{\mathrm{Re}(\beta-\alpha)+ik\mathrm{Im}(\beta-\alpha)} \\ = & \ [U(1)-U(0)]+i[V(1)-V(0)] \\ = & \ U'(t_1)+iV'(t_2) \\ = & \ \mathrm{Re}f'_{(k)}[\alpha+t_1(\beta-\alpha)]+i\mathrm{Im}f'_{(k)}[\alpha+t_1(\beta-\alpha)]. \end{aligned}$$

Let $\xi=\alpha+t_1(\beta-\alpha)$, $\eta=\alpha+t_2(\beta-\alpha)$. We obtain that

$$\frac{f(\beta)-f(\alpha)}{\mathrm{Re}(\beta-\alpha)+ik\mathrm{Im}(\beta-\alpha)}=\mathrm{Re}f'_{(k)}(\xi)+i\mathrm{Im}f'_{(k)}(\eta).$$

□

From Theorem 2.3, we have

Corollary 2.1. *Let $f(z)$ be a K-analytic on domain D, line segment $C(\alpha,\beta)\subset D$, $f(\alpha)=f(\beta)$. Then there exist $\xi,\ \eta\in C[\alpha,\beta]$, such that*

$$\mathrm{Re}f'_k(\xi)=\mathrm{Im}f'_{(k)}(\eta)=0.$$

3 Conclusion

In the present paper, we extend differential and integral mean value theorems from analytic and conjugate analytic functions to K-analytic functions.

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Competing Interests

Authors have declared that no competing interests exist.

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