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Sinc-collocation Algorithm for Solving Nonlinear Fredholm Integro-differential Equations

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Abstract

This paper is concerned with the numerical solution using sinc-collocation method of nonlinear Fredholm integro-differential equations. The algorithm is based on replacing the exact solution by a linear combination of sinc functions. The resulting nonlinear equations are treated using Newton method. Numerical examples illustrate the pertinent features of the method and its applicability to a large variety. The examples include convolution type, singular as well as singularly-perturbed problems.

Keywords: Sinc function, Collocation method, Nonlinear Fredholm integral equation, Integro-differential equation.

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1 Introduction

Nonlinear phenomena, that appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modelled by partial differential equation, integral equations or by integro-differential equations.

Nonlinear integro-differential equations are usually hard to solve analytically and exact solutions are scarce. Therefore, they are treated numerically or semi analytic-numerical methods are used. Previous treatments of these equations include: The method of upper and lower solutions was used to study some problems [1,2,3]. Use of Taylor series [4] and computer algebra for nonlinear Volterra-Fredholm integro-differential equations in [5]. Adomian's decomposition [6,7] was employed to treat coupled nonlinear system of Fredholm integro-differential equations. In [8] Adomian decomposition

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was applied to a first order special nonlinear Fredholm integro-differential in two variables. Optimal order spline methods were used in [9]. Neta [10] employed Galerkin's method to a special first order nonlinear Fredholm integro-differential equation in two unknowns. In [11] orthogonal collocation was applied to a second order Fredholm integro-differential equation. Homotopy perturbation was employed in [12,13]. Sine-cosine wavelet-Galerkin was used in [14]. Maleknejad and Nedaiasl [15] employed sinc-collocation method to a class of nonlinear Fredholm integral equation. Also, Haar wavelet method was applied in [16].

The technique that we used is the sinc-collocation method, which is based on the Whittaker-Shannon-Kotel'nikov sampling theorem for entire functions. It is introduced by F. Stenger more than twenty years ago [17]. The use of sinc functions has received much attention from both mathematicians and engineers alike because of the comprehensive mathematical power and the good application potentials of sinc function in many interesting problems. This approach, which uses entire functions as bases, offers ease of implementation and accurate approximation, even in the presence of singularities and has many advantages over classical methods that use polynomials as bases. It gives a much better rate of convergence and accuracy than polynomial methods. Indeed, the sinc method is a powerful alternative for the numerical solution of both integral and differential equations.

This paper describes a novel procedure for solving nonlinear Fredholm integro-differential equations, based on sinc methods. The method's implementation requires no modification in the presence of singularities. The approximating discrete system depends only on parameters of the differential equation regardless of whether it is singular or nonsingular. The goals of this method are twofold. We first aim to confirm that the sinc-collocation method is powerful, efficient, and promising in handling these problems, linear and nonlinear as well. We second aim to support that our work yields accurate results for illustrative nonlinear problems and comparison with homotopy perturbation method (HPM) is made.

This paper is a continuation of the previous work of the authors [18,19] to develop sinc-collocation method for the numerical computations of nonlinear Fredholm integro-differential equations.

The organization of the paper is as follows. In Section 2, we describe the basic formulation of sinc functions required for our subsequent development. Section 3 is devoted to derivation of the discrete system. In Section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2 Sinc Function

In this study, sinc-collocation method is developed for nonlinear second order Fredholm integro-differential boundary value problem in the form

$$\begin{gathered}\sum_{i=0}^{2}\mu_i(x)\,u^{(i)}(x)=f(x)+\lambda\int_a^b K(x,t)\,u^n(t)\,dt,\qquad x\in J=[a,b]\\ u(a)=\gamma\qquad u(b)=\beta\end{gathered}\tag{2.1}$$

where $K(x,t),\ f(x)$, $u(x)$ and $\mu_i(x)$, are analytic functions, λ is a parameter,n is integer and γ and β are real constant. It will always be assumed that (2.1) possesses a unique solution $u\in C^n(J)$.

The sinc-collocation procedure for solving the problem (2.1) begins by selecting composite sinc functions appropriate to the intervals (a,b) so that their translates form basis functions for the expansion of the approximate solution $u(x)$. A through review of properties of the sinc function and the general sinc-Collocation method can be found in [17,20].

The rest of this section contains an overview of properties of the sinc function that are used in the sequel.

If $f(x)$ is defined on the real line, then for $h > 0$ the Whittaker cardinal expansion of f

$$f_m(x) = \sum_{k=-N}^{N} f_k \, S(k,h) \circ \phi(x), \quad m = 2\,N + 1$$

where $f_k = f(x_k)$, $x_k = hk$ and the mesh size is given by

$$h = \sqrt{\frac{2\,\pi\,d}{\alpha\,N}}, \qquad 0 < \alpha \leq 1, \qquad d \leq \frac{\pi}{2}$$

where N is suitably chosen and α depends on the asymptotic behavior of $f(x)$. The basis functions on (a,b) are then given by

$$S(k,h) \circ \phi(x) = \text{sinc}\left(\frac{\phi(x) - k\,h}{h}\right)$$

and

$$\phi(x) = \ln\left(\frac{x-a}{b-x}\right) \tag{2.2}$$

The interpolation formula for $f(x)$ over $[a,b]$ takes the form

$$f(x) \approx \sum_{k=-N}^{N} f_k \; S(k,h) \circ \phi(x), \tag{2.3}$$

where

$$f_k = f(x_k), \quad \text{and} \quad x_k = \frac{a + be^{kh}}{1 + e^{hk}}.$$

The n-th derivative of the function f at points x_k can be approximated using a finite number of terms as

$$f^{(n)}(x) \approx \sum_{k=-N}^{N} f_k \frac{d^n}{d\,x^n} \; [S(k,h) \circ \phi(x)]\,. \tag{2.4}$$

Integral of $f(x)$ is given by

$$\int_a^b f(x)\,d\,x \approx h \sum_{k=-N}^{N} \frac{f_k}{\phi'(x_k)}, \tag{2.5}$$

Let

$$\frac{d^i}{d\phi^i}[S(j,h) \circ \phi(x)] = S_j^{(i)}(x), \qquad 0 \leq i \leq 2, \tag{2.6}$$

we note

$$\begin{aligned} \frac{d}{dx}[S(j,h) \circ \phi(x)] &= S_j^{(1)}(x)\,\phi'(x) \\ \frac{d^2}{dx^2}[S(j,h) \circ \phi(x)] &= S_j^{(2)}(x)\;\left[\phi'(x)\right]^2 + S_j^{(1)}(x)\,\phi''(x). \end{aligned} \tag{2.7}$$

and

$$\delta_{j\,k}^{(n)} = h^n \, \frac{d^n}{d\,\phi^n} \; [S(j,h) \circ \phi(x)]_{x=x_k}$$

which will be used later in Theorem 3.2.

3 The Sinc-Collocation Method

We assume that $u(x)$, the solution of (2.1), is approximated by the finite expansion of sinc basis functions

$$u_m(x) = \sum_{j=-N}^{N} u_j \, S(j,h) \circ \phi(x), \qquad m = 2\,N + 1. \tag{3.1}$$

Application of equation (2.5) to the kernel integral in (2.1) gives the following lemma

Lemma 3.1. *The following relation holds*

$$\int_a^b K(x,t)\, u^n(t)\, d\,t \approx h \sum_{j=-N}^{N} \frac{K(x,t_j)}{\phi'(t_j)}\, u_j^n, \tag{3.2}$$

where u_j denotes an approximate value of $u(x_j)$.

If we replace the second term on the right-hand side of (2.1) with the right-hand side of (3.2) we have

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{2} \mu_i(x)\, \frac{d^i}{d\,x^i}\, S(j,h) \circ \phi(x) \right] u_j - h\,\lambda \sum_{j=-N}^{N} \frac{K(x,t_j)}{\phi'(t_j)}\; u_j^n = f(x). \tag{3.3}$$

Using (2.6)and (2.7), and substituting $x = x_k = \phi(k\,h)$ in (3.3) and applying the collocation method to it, we eventually obtain the following theorem

Theorem 3.2. *If the assumed approximate solution of problem (2.1) is (3.1), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{u_j, -N < j < N\}$ is given by*

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{2} g_i(x_k)\, \frac{\delta_{k\,j}^{(i)}}{h^i} \right] u_j - h\,\lambda \sum_{j=-N}^{N} \frac{K(x_k,t_j)}{\phi'(t_j)}\, u_j^n = f_k, \quad k = -N, -N+1, \ldots, N \tag{3.4}$$

where

$$g_0(x_k) = \mu_0(x_k), \qquad g_2(x_k) = \mu_2(x_k)\; \left[\phi'(x_k)\right]^2,$$
$$g_1(x_k) = \mu_1(x_k)\,\phi'(x_k) + \mu_2(x_k)\,\phi''(x_k).$$

To obtain a matrix representation of the equations in (3.4), recall the notation of Toeplitz matrices [21]. We note that

$$\delta_{k\,j}^{(0)} = \delta_{j\,k}^{(0)}, \quad \delta_{k\,j}^{(2)} = \delta_{j\,k}^{(2)} \quad \text{and} \quad \delta_{k\,j}^{(1)} = -\delta_{j\,k}^{(1)}.$$

Let $\mathbf{D}(g(x_j))$ denote the $m \times m$ diagonal matrix with

$$\mathbf{D}(g(x))_{i\,j} = \begin{cases} g(x_i) & i = j, \\ 0 & i \neq j. \end{cases}$$

Let $\mathbf{u}$ be the m-vector with j-th component given by u_j , and let $\mathbf{u}^n$ be the m-vectors with j-th component given by u_j^n and $\mathbf{1}$ is an m-vector each of whose components is 1. In this notation the system in (3.4) takes the matrix form

$$\mathbf{A}\,\mathbf{u} + \mathbf{B}\,\mathbf{u}^n = \Theta, \tag{3.5}$$

where

$$\Theta = \mathbf{D}\,(f)\;\mathbf{1},$$
$$\mathbf{u} = [u_{-N}, u_{-N+1}, \ldots, u_N]^{\tau},$$

$$\mathbf{A} = \sum_{i=0}^{n} \frac{1}{h^i} \mathbf{I}^{(i)} \mathbf{D}(g_i).$$

$$\mathbf{B} = -h\,\lambda\, \frac{K(x_k, t_j)}{\phi'(t_j)},$$

and

$$\mathbf{I}^{(i)} = \left[\delta_{kj}^{(i)}\right], \quad \text{for} \quad i = 0, 1, 2.$$

Now we have a nonlinear system of m equations for the m unknown coefficients, namely, $\{u_j\}_{j=-N}^{N}$. We can obtain the coefficients of the approximate solution by solving this nonlinear system by *Newton's method*, see [22].

4 Numerical Examples

Three examples are given to illustrate the performance of our method. All the experiments are performed in MATLAB with a machine precision of 10^{-16}. In our tests, the zero vector is the initial guess and the stopping criterion is

$$\|\mathbf{u}_{j+1} - \mathbf{u}_j\| \le 10^{-7}.$$

In all the examples we take $d = \pi/2$, $\lambda = 1$, $a = 0$, $b = 1$, and $\alpha = 1$. Moreover, all the problems have homogeneous Dirichlet boundary conditions and known solutions.

The maximum absolute error between the numerical approximation and the exact solution at the sinc grid points is determined and reported as

$$\|E_s\| = \max_{-N \le i \le N} \left| u_{\text{exact solution}}(x_i) - U_{\text{sinc-collocation}}(x_i) \right|,$$

where

$$x_i = \frac{a + b\,e^{i\,h}}{1 + e^{i\,h}}.$$

Example 1: For the sake of comparison, we consider the same problem discussed by Biazar,and Ghazvini [23] , who used the homotopy perturbation method to obtain their numerical solution. Consider the Fredholm integral equation

$$u(x) = f(x) + \frac{1}{2}\int_0^1 (x - t)\,u^2(t)\,dt, \quad 0 < x < 1,$$

$$f(x) = x\,\ln\,x - \frac{53}{108}\,x + \frac{1}{3}\,\ln\,2\,\left(\frac{8}{3}\,x + 2 - x\,\ln\,2\right) - \frac{241}{576}$$

whose exact solution is

$$u(x) = x\,\ln(x + 1).$$

Maximum absolute error is tabulated in **Table 4.1** for sinc-collocation together with the corresponding results of Biazar,and Ghazvini [23].

Table 4.1 Maximum absolute error for Example 1

sinc-collocation method for $N = 100$	The homotopy perturbation method [23]
1.3245 E-010	1.80788 E-007

Example 2: [13] Consider the following nonlinear Fredholm integro-differential equations

$$u' = 1 - \frac{x}{4} + \int_0^1 x\,t\,u^2(t)\,dt, \quad 0 < x < 1,$$

and subject to the boundary conditions

$$u(0) = 0,$$

then the exact solution is

$$u(x) = x.$$

Maximum absolute error is tabulated in **Table 4.2** for sinc-collocation together with the corresponding results of He's homotopy perturbation method [13] .

Table 4.2 Maximum absolute error for Example 2

sinc-collocation method for $N = 100$	The homotopy perturbation method [13]
3.4564 E-09	2.5739 E-005

Example 3: In the case, $\mu_1(x) = 1/x, \quad \mu_0(x) = 1/x^2$ and $n = 3$ equation (2.1) becomes

$$u'' + \frac{1}{x}\,u' + \frac{1}{x^2}\,u = f(x) + \int_0^1 K(x,t)\,u^3(t)\,dt, \quad 0 < x < 1,$$

If

$$f(x) = -5 + \frac{2}{x} - x \quad \text{and} \quad K(x,t) = \frac{30\,x}{t - t^2}$$

and subject to the boundary conditions

$$u(0) = 0 \quad u(1) = 0,$$

then the exact solution is

$$u(x) = x\,(1 - x).$$

The maximum absolute error, $\|E_s\|$, is reported in **Table 4.3** as N increases from $N = 10$ to $N = 60$.

Table 4.3 $\|E_s\|$ for Example 3

N	$\|E_s\|$
10	9.9891 E-005
20	1.6552 E-006
30	1.3102 E-007
40	1.1374 E-008
50	2.9187 E-009
60	1.5782 E-010

5 Conclusion

The feasibility of the sinc-collocation method in the numerical solutions of nonlinear integro-differential equation is investigated. The details of the discretization process is demonstrated and the corresponding approach is developed. The presented calculations show that this method is effective for solving such problems. The results play an essential role in extending our method to solve general integro-differential equations.

In future, since this method is relatively easy to implement and computationally inexpensive, we would like to extend it to partial integro-differential equations.

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Competing Interests

The authors declare that no competing interests exist.

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