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Energy Decay Rate for Bresse System with Nonlinear Localized Damping

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Abstract

In this paper, we study the energy decay rate for the Bresse system in a one-dimensional bounded domain with nonlinear localized damping acting in all the three wave equations. We show the asymptotic stability without impose conditions about the equal-speed wave propagation using a method developed by Kormornik [1994] and Martinez [1999], providing a larger class for non-linear functions.

Keywords: Bresse system, localing damping, energy decay, Komornik's inequality.

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1 Introduction

We consider the initial-boundary value problem for the Bresse system with weak nonlinear localized dissipation

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + \varphi + \alpha_1(x)g_1(\varphi_t) = 0\,, \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \alpha_2(x)g_2(\psi_t) = 0\,, \tag{1.2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \alpha_3(x)g_3(w_t) = 0\,, \tag{1.3}$$

in $(0,L)\times R^+$, where $L>0$ is a constant and $R^+ = (0,+\infty)$, with Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, t\in R^+, \tag{1.4}$$

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and initial conditions

$$\varphi(\cdot,0)=\varphi_0, \psi(\cdot,0)=\psi_0, w(\cdot,0)=w_0, \tag{1.5}$$

$$\varphi_t(\cdot,0)=\varphi_1, \psi_t(\cdot,0)=\psi_1, w_t(\cdot,0)=w_1. \tag{1.6}$$

Here $\rho_1, \rho_2, b, l, k, k_0$ are positive constants which related to composition of the material, $g_i : R \to R, i = 1,2,3$, are continuous nondecreasing function and $\alpha_i, i = 1,2,3$, are positive functions. By w, φ and ψ we are denoting, respectively, the longitudinal, vertical and shear angle displacements, and $\{\varphi, \psi, w\}$ is a sought solution of (1.1)-(1.3). Elastic structures of the arches type are objects of study in many areas like mathematics, physics and engineering. For more details, the interested reader can visit the works of Liu and Rao[1]; Boussouira et al.[2] and references therein.

There exist a few results about the stability of the Bresse system where the authors consider the different kinds of the dissipative mechanism. For example, in the works of Liu and Rao [1], they consider the Bresse system with two different dissipative mechanisms, given by two heat equations, non-dissipative coupled to the system. Boussouira et al. [2]; Noun and Wehbe [3] proved that the semi group associated with the Bresse system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \tag{1.7}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\varphi_t = 0, \tag{1.8}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \tag{1.9}$$

with boundary conditions of the Dirichlet-Dirichlet-Dirichlet type or mixed boundary conditions is polynomially stable provided

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}, \text{and } k = k_0$$

(i.e., the equal-speed wave propagation condition) and moreover they proved the lack of exponential stability when they considered the Dirichlet-Neumann-Neumann type boundary condition. The equal-speed wave propagation condition has been used in many works in order to establish exponential decay rates, see for instance Boussouira et al. [2]; Noun and Wehbe [3]; Fatori and Monteiro [4]; Soriano et al. [5]; Fatori and Rivera [6]. Fatori and Monteiro [4] showed the optimality of the polynomial decay rate for the Bresse system (1.7)-(1.9) with the Dirichlet-Neumann-Neumann type boundary condition. Soriano et al. [5] proved the asymptotic stability for the system with indefinite damping mechanism.

Wehbe and Youssef [7]; Santos and Junior [8] showed the asymptotic stability without impose conditions about the equal-speed wave propagation for the system for the Bresse system with linear dissipation by different methods.

Soriano et al. [9] and Charles et al. [10] gave the asymptotic stability for the following Bresse system with nonlinear dissipation

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + \alpha_1(x)g_1(\varphi_t) = 0, \tag{1.10}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \alpha_2(x)g_2(\psi_t) = 0 \tag{1.11}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \alpha_3(x)g_3(w_t) = 0 \tag{1.12}$$

by energy methods. However, to obtain the energy decay rate estimate, the author required that α_i and the damping terms $g_i(\cdot)$ satisfy the following growth rate:

$$\begin{aligned}&\alpha_i = \alpha_i(x) \in L^\infty(0, L), \alpha_i(x) \geq c > 0,\\ &g_i(s)s > 0, \text{ for } s \neq 0, cs \leq g_i(s) \leq ds \text{ for } |s| > 1, i = 1, 2, 3.\end{aligned} \tag{1.13}$$

where C, c, d are constant.

Fatori and Rivera [6] consider the Bresse system (1.7)-(1.9) with thermal dissipation effective in additional equation of the system. There are some results about the stability and global attractors for thermoelastic Bresse system (see Ma[11]; Han and Xu [12]).

Our main goal is to obtain rates of decay for the nonlinear localized damped system (1.1)-(1.6) by using the multiplier method. In this paper, inspired by the works of Komornik [13] and Martinez [14], we extend the behavior of $\alpha_i, g_i(\cdot)$ to more general cases which does not necessarily satisfy (1.13) and get the explicit energy decay rate estimate for the system (1.1)-(1.6).

Adopting the methods of the works Komornik [13]; Martinez [14]; Guesmia[15]; Haraux and Zuazua [16], we construct an energy functional which is equivalent to the energy of the problem (1.1)-(1.6) and then prove that the functional satisfies a differential inequality from which our energy decay rate estimate can be established.

There is much literature concerned with the energy decay rate estimates for related problems, for more recent results we refer the reader to Benaissa and Mokeddem [17]; Burio [18]; Cavalcanti et al. [19]; Zhang and Zuazua [20].

Our paper is organized as follows. In Section 2 we present the main assumptions and the existence and uniqueness of solutions of the problem (1.1)-(1.6). Section 3 is devoted to the proof of the main result.

2 Preliminaries and the Existence and Uniqueness of Solutions

In this section, we present some material needed for the proof of our results and the existence and uniqueness of solutions by Galerkin method. Throughout this paper, we denote by

$\|\cdot\|, \|\cdot\|_p, \|\cdot\|_{H_0^1}$ the usual norms in space $L^2(0,L), L^p(0,L)$ and $H_0^1(0,L)$, respectively, and $(u,v) = \int_\Omega u(x)v(x)dx$, $Q = (0,L)\times(0,T)$.

We suppose that $\alpha_i(x)$ and $g_i(s)$ satisfy the following hypotheses:

(i) $\alpha_i = \alpha_i(x) \in L^\infty(0,L), \alpha_i(x) \ge c > 0, i = 1,2,3.$

(ii) $g_i(t): R \to R$ is a nondecreasing C^0 function and suppose that there exist constant $K_2 \ge K_1 > 0$ such that for $i = 1,2,3$

$$g_i(s)s > 0 (s \ne 0),\ K_1 min\{|s|, |s|^p\} \le |g_i(s)| \le K_2 max\{|s|, |s|^{\frac{1}{p}}\},\ p \ge 1.$$

We first state a well-known lemma that will be needed later.

Lemma 2.1 (Komornik [13]; Martinez [14]) Let $E: R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $p \ge 1$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}} dt \le AE(S), 0 \le S < +\infty,$$

then

$$E(t) \le cE(0)(1+t)^{-\frac{2}{p-1}}, t \ge 0, \text{ if } p > 1,$$

$$E(t) \le cE(0)e^{-\omega t}, t \ge 0, \text{ if } p = 1.$$

where c and ω are positive constants independent of the initial value $E(0)$.

If w, φ and ψ is a solution of (1.1)-(1.6) then the energy of system related to this solution will be denoted by $E(t)$, with t nonnegative, and given by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + b|\varphi_x|^2 + |\varphi|^2 + k_0|w_x - l\varphi|^2 + k|\varphi_x + \psi + lw|^2\right)dx. \quad (2.1)$$

We can prove that the system (1.1)-(1.6) is dissipative as stated below:

Lemma 2.2 (Charies et al. [7]; Soriano et al. [8]) The energy functional $E(t)$ defined by (2.1), satisfies:

$$\frac{d}{dt}E(t) = -\int_0^L \{\alpha_1(x)g_1(\varphi_t)\varphi_t + \alpha_2(x)g_2(\psi_t)\psi_t + \alpha_3(x)g_3(w_t)w_t\}dt \le 0. \quad (2.2)$$

There are many results about the global existence by virtue of the semi-group arguments (see Soriano et al. [5]; Boussouira et al. [2]; Charies et al. [7]; Fatori and Rivera [6]; Fatori and

Monteiro [4]; Soriano et al. [8]). Next, for completeness, we state and present a brief discussion of the existence, uniqueness of the solutions of (1.1)-(1.6) by standard Galerkin method.

Theorem2.3 Let $\varphi_0, \psi_0, w_0 \in H_0^1$ and φ_1, ψ_1, $w_1 \in L^2$, and assume that $g_i, \alpha_i, i = 1,2,3,$ satisfy the above condition. Then the problem (1.1)-(1.6) admits a uniqueness weak solution (φ, ψ, w) such that

$$\varphi,\ \psi,\ w \in L^\infty\left(0,T;H_0^1(\Omega)\right);\ \varphi_t,\ \psi_t,\ w_t \in L^\infty\left(0,T;L^2\right)$$

and satisfies the equation (1.1)-(1.3) in the weak sense and the initial condition (1.5)-(1.6).

Proof Let $\{v_j\}$ be orthonormal bases in $H_1^0 \cap L^2$ and denote $V_m = span\{v_1, v_2, \cdots, v_m\}$. We find the approximate solution of problem (1.5)-(1.6) in the form

$$\varphi_m = \sum_{j=1}^m f_{jm}(t) v_j; \psi_m = \sum_{j=1}^m h_{jm}(t) v_j; w_m = \sum_{j=1}^m r_{jm}(t) v_j,$$

where the coefficient functions $f_{jm}(t), h_{jm}(t), r_{jm}(t)$ are solutions to the approximate problem

$$\rho_1(\varphi_{mtt}, v) + k(\varphi_{mx} + \psi_m + lw_m, v_x) - k_0 l(w_{mx} - l\varphi_m + \varphi, v) + (\alpha_1(x) g_1(\varphi_{mt}), v) = 0, \tag{2.3}$$

$$\rho_2(\psi_{mtt}, v) + b(\psi_{mx}, v_x) + k(\varphi_{mx} + \psi_m + lw_m, v) + (\alpha_2(x) g_2(\psi_{mt}), v) = 0, \tag{2.4}$$

$$\rho_1(w_{mtt}, v) + k_0(w_{mx} - l\varphi_m, v_x) + kl(\varphi_{mx} + \psi_m + lw_m, v) + (\alpha_3(x) g_3(w_{mt}), v) = 0, \tag{2.5}$$

$$\varphi_m(0) = \varphi_{0m}, \psi_m(0) = \psi_{0m}, w_m(0) = w_{0m}, \tag{2.6}$$

$$\varphi_{mt}(0) = \varphi_{1m}, \psi_{mt}(0) = \psi_{1m}, w_{mt}(0) = w_{1m}, \tag{2.7}$$

for any $v \in V_m$. Here $\varphi_{0m} = \sum_{i=1}^m (\varphi_0, v_j) v_j \to \varphi_0, \psi_{0m} = \sum_{i=1}^m (\psi_0, v_j) v_j \to \psi_0, w_{0m} = \sum_{i=1}^m (\varphi_0, v_j) v_j \to w_0$ strongly in H_0^1 ; $\varphi_{1m} = \sum_{i=1}^m (\varphi_1, v_j) v_j \to \varphi_1, \psi_{1m} = \sum_{i=1}^m (\psi_1, v_j) v_j \to \psi_1, w_{1m} = \sum_{i=1}^m (\varphi_1, v_j) v_j \to w_1$ strongly in L^2.

From the assumptions of Theorem 2.3, system (2.3)-(2.7) has a local solution φ_m, ψ_m, w_m, on some interval $[0, T_m]$ with $0 < T_m < T$ and we can extend this solution to whole interval $[0,T]$ by making use of a priori estimates below.

Putting $v = \varphi_{mt}(t), v = \psi_{mt}(t)$ and $v = w_{mt}(t)$ in (2.3), (2.4) and (2.5), respectively, and adding these three equations, we obtain

$$E_m(t)+\int_0^t\int_0^l\{\alpha_1(x)g_1(\varphi_{mt})\varphi_{mt}+\alpha_2(x)g_2(\psi_{mt})\psi_{mt}+\alpha_3(x)g_3(w_{mt})w_{mt}\}dxdt=E_m(0), \quad (2.8)$$

where

$$E_m(t)=\frac{1}{2}\int_0^L\Big(\rho_1|\varphi_{mt}|^2+\rho_2|\psi_{mt}|^2+\rho_1|w_{mt}|^2+b|\varphi_{mx}|^2+|\varphi_m|^2$$

$$+k_0 l|w_{mx}-l\varphi_m|^2+k|\varphi_{mx}+\psi_m+lw_m|^2\Big)dx, \quad (2.9)$$

which implies that $E_m(t)$ is non-increasing in $[0,T_m)$ and then $E_m(t)\le E_m(0)$ .Therefore, employing the assumptions about φ_0, ψ_0, w_0, we deduce that there exists a constant $C=C(T)>0$, independent of m, and $t\in[0,T_m)$, such that

$$\int_0^L\Big(\rho_1|\varphi_{mt}|^2+\rho_2|\psi_{mt}|^2+\rho_1|w_{mt}|^2+b|\varphi_{mx}|^2+|\varphi_m|^2$$

$$+k_0 l|w_{mx}-l\varphi_m|^2+k|\varphi_{mx}+\psi_m+lw_m|^2\Big)dx\le C. \quad (2.10)$$

The above estimates allow one to take $T_m=T$ for all m and gives

$$\varphi_m,\psi_m,w_m \text{ is bounded in } L^\infty\left(0,T;H_0^1\right);$$

$$\varphi_{mt},\psi_{mt},w_{mt} \text{ is bounded in } L^\infty\left(0,T;L^2\right)\cap L^p(Q).$$

Then by the compactness lemma and the embedding $H_0^1\subset L^r, 1\le r<+\infty$, there exist a subsequence of φ_m, a subsequence of ψ_m, a subsequence of w_m, a subsequence of φ_{mt}, a subsequence of ψ_{mt} and a subsequence of w_{mt} which we still denote by the same notations, and functions φ, ψ, w such that

$$\varphi_m\to\varphi,\psi_m\to\psi,w_m\to w \text{ weak * in } L^\infty\left(0,T;H_0^1\right);$$

$$\varphi_{mt}\to\varphi_t,\psi_{mt}\to\psi_t,w_{mt}\to w_t \text{ weak * in } L^\infty\left(0,T;L^2\right) \text{ and weak * in } L^p(Q);$$

$$\varphi_m\to\varphi,\psi_m\to\psi,w_m\to w \text{ a.e. strongly in } L^2(Q).$$

Analogous arguments as in Section 6.2 of Lions [21], we have

$$g_1(\varphi_{mt})\to g_1(\varphi_t),g_2(\psi_{mt})\to g_2(\psi_t),g_3(w_{mt})\to g_3(w_t) \text{ weak in } L^{p'}(Q),$$

and then letting $m \to \infty$ in (2.3), (2.4) and (2.5) for any $v \in H_0^1$, we established the existence and uniqueness of the weak solution.

Furthermore, we can get the following result about regular solution but we omit the proof.

Theorem 2.4 Let $\varphi_0, \psi_0, w_0 \in H^2 \cap H_0^1$ and $\varphi_1, \psi_1, w_1 \in L^2 \cap H_0^1$, and assume that g_i, α_i, $i = 1,2,3,$ satisfy the above condition. Then the problem (1.1)-(1.6) admits a uniqueness regular solution (φ, ψ, w) such that

$$\varphi, \psi, w \in L^\infty\left(0,T; H^2 \cap H_0^1\right); \varphi_t, \psi_t, w_t \in L^\infty\left(0,T; L^2 \cap H_0^1\right); \varphi_{tt}, \psi_{tt}, w_{tt} \in L^\infty\left(0,T; L^2\right)$$

and satisfies the equation (1.1)-(1.3) in the weak sense and the initial condition (1.5)-(1.6).

3 Energy Decay of Solution

In this section, we prove our decay result. The definition and the existence of solutions of (1.1)-(1.6) are the same as Theorem 2.3.

Theorem 3.1 Assume that the condition about $\alpha_i(x), g_i(t), i = 1,2,3,$ hold. Let $\varphi_0, \psi_0, w_0 \in H_0^1$ and $\varphi_1, \psi_1, w_1 \in L^2$. Then the solution of the problem (1.1)-(1.6) satisfies

$$E(t) \le cE(0)e^{-\omega t}, t \ge 0, p = 1,$$

$$E(t) \le cE(0)(1+t)^{-\frac{2}{p-1}}, t \ge 0, p > 1,$$

where c and ω are positive constants independent of the initial value $E(0)$.

Proof We multiply equation (1.1) by $E(t)^{\frac{p-1}{2}}\varphi$, equation (1.2) by $E(t)^{\frac{p-1}{2}}\psi$ and equation (1.3) by $E(t)^{\frac{p-1}{2}}w$, and then integrate these three equations $(0,L)\times(S,T)$, we have

$$0 = \int_S^T E(t)^{\frac{p-1}{2}} \int_0^L \left[\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - kl(w_x - l\varphi) + \varphi + \alpha_1(x) g_1(\varphi_t)\right]\varphi dxdt$$

$$= \left(E(t)^{\frac{p-1}{2}} \int_0^L \rho_1\varphi_t\varphi dx\right)\Bigg|_S^T - \frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}} E'(t)\int_0^L \rho_1\varphi_t\varphi dxdt + k\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \varphi_x^2 dxdt$$

$$+\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \left[\left(-k(\psi + lw)_x - k_0 l(w_x - l\varphi) + \varphi + \alpha_1(x) g_1(\varphi_t)\right)\varphi - \rho_1\varphi_t^2\right]dxdt, \tag{3.1}$$

$$0 = \int_S^T E(t)^{\frac{p-1}{2}} \int_0^L \left[\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \alpha_2(x) g_2(\psi_t)\right]\psi dxdt$$

$$=\left(E(t)^{\frac{p-1}{2}}\int_0^L \rho_2\psi_t\psi dx\right)\Bigg|_S^T-\frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}}E'(t)\int_0^L \rho_2\psi_t\psi dxdt+b\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \psi_x^2 dxdt$$

$$+\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\Big[\big(k(\varphi_x+\psi+lw)+\alpha_2(x)g_2(\psi_t)\big)\psi-\rho_2\psi_t^2\Big]dxdt, \tag{3.2}$$

$$0=\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\Big[\rho_1 w_{tt}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)+\alpha_3(x)g_3(w_t)\Big]wdxdt$$

$$=\left(E(t)^{\frac{p-1}{2}}\int_0^L \rho_1 w_t w dx\right)\Bigg|_S^T-\frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}}E'(t)\int_0^L \rho_1 w_t w dxdt+k_0\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L w_x^2 dxdt$$

$$+\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\Big[\big(k_0 l\varphi_x+l(\varphi_x+\psi+lw)+\alpha_3(x)g_3(w_t)\big)w-\rho_1 w_t^2\Big]dxdt, \tag{3.3}$$

Taking the sum of the above three equations, we get

$$2\int_S^T E(t)^{\frac{p+1}{2}}dt=\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\Big(\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\rho_1|w_t|^2+b|\psi_x|^2+k_0 l|w_x-l\varphi|^2+k|\varphi_x+\psi+lw|^2\Big)dxdt$$

$$=\left(E(t)^{\frac{p-1}{2}}\int_0^L(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi+\rho_1 w_t w)dx\right)\Bigg|_S^T$$

$$+\frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}}E(t)'\int_0^L(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi+\rho_1 w_t w)dxdt$$

$$-\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\big(\alpha_1(x)\varphi g_1(\varphi_t)+\alpha_2(x)\psi g_2(\psi_t)+\alpha_3(x)wg_3(w_t)\big)dxdt$$

$$+2\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\big(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2\big)dxdt\,. \tag{3.4}$$

Noting that $\|\psi\|\le C_1\|\psi_x\|$ and

$$\|\varphi\|\le C_1\|\varphi_x+\psi+lw\|+\|\psi\|+l\|w\|,\|w\|\le C_1\|w_x-l\varphi\|+l\|\varphi\|, \tag{3.5}$$

since $E(t)$ is non-increasing and non-negative function on R^+, using Holder's inequality, Poincare's inequality, Young's inequality and the expressions of $E(t)$, we have

$$\left(E(t)^{\frac{p-1}{2}}\int_0^L(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi+\rho_1 w_t w)dx\right)\Bigg|_S^T\le CE(T)^{\frac{P-1}{2}}\big(E(T)+E(S)\big)\le CE(S)^{\frac{P+1}{2}}, \tag{3.6}$$

$$\frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}} E'(t)\int_0^L (\rho_1\varphi_t\varphi+\rho_2\psi_t\psi+\rho_1 w_t w)dxdt$$

$$\le C\int_S^T E(t)^{\frac{P-1}{2}} E'(t)dt \le CE(S)^{\frac{P-1}{2}}\left|\int_S^T E'(t)dt\right| \le CE(S)^{\frac{P+1}{2}}, \tag{3.7}$$

where and in the following C denote the different constants. Using (3.6) and (3.7), we conclude from (3.4) that

$$2\int_S^T E(t)^{\frac{p+1}{2}}dt \le cE(S)+c\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L\Big[\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\rho_1|w_t|^2$$

$$+\alpha_1(x)\varphi g_1(\varphi_t)+\alpha_2(x)\psi g_2(\psi_t)+\alpha_3(x)wg_3(w_t)\Big]dxdt\,. \tag{3.8}$$

Now, we estimate the terms of the right-hand side of (3.8) in order to apply the results of Lemma 2.1. By the assumption about $\alpha_i(x)$, the condition (ii) about g_i, Sobolev embedding, Holder's inequality, (3.5), (2.2) and the expressions of $E(t)$, we get

$$\left|-\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \alpha_1(x)\varphi g_1(\varphi_t)dxdt\right| \le C\int_S^T E(t)^{\frac{p+1}{2}}dt + C\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L |g_1(\varphi_t)|^2dxdt$$

$$\le CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left[\int_{|\varphi_t|>1}|g_1(\varphi_t)|^2dx+\int_{|\varphi_t|\le 1}|g_1(\varphi_t)|^2dx\right]dt$$

$$\le CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left[\int_{|\varphi_t|>1}g_1(\varphi_t)\cdot\varphi_t dx+\int_{|\varphi_t|\le 1}\left(g_1(\varphi_t)\cdot\varphi_t\right)^{\frac{2}{p+1}}dx\right]dt$$

$$\le CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(\int_{|\varphi_t|\le 1}\left(g_1(\varphi_t)\cdot\varphi_t\right)dx\right)^{\frac{2}{p+1}}dt$$

$$\le CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt\,. \tag{3.9}$$

We may prove in the same way the following estimate

$$\left|-\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \alpha_2(x)\psi g_2(\psi_t)dxdt\right|$$

$$\le CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt\,, \tag{3.10}$$

and

$$\left|-\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \alpha_3(x)wg_3(w_t)dxdt\right|$$

$$\leq CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt \ .\quad (3.11)$$

By the assumption about the condition (ii) about g_i and (2.2), we get

$$\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \rho_1\varphi_t^2 dxdt$$

$$=\int_S^T E(t)^{\frac{p-1}{2}}\left[\int_{|\varphi_t|\leq 1}\rho_1\varphi_t^2dx+\int_{|\varphi_t|>1}\rho_1\varphi_t^2dx\right]dt$$

$$\leq C\int_S^T E(t)^{\frac{p-1}{2}}\left[\int_{|\varphi_t|>1}g_1(\varphi_t)\cdot\varphi_t dx+\int_{|\varphi_t|\leq 1}(\varphi_t)^{\frac{2}{p+1}}(\varphi_t)^{\frac{2p}{p+1}}dx\right]dt$$

$$\leq C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(\int_{|\varphi_t|\leq 1}\left(g_1(\varphi_t)\cdot\varphi_t\right)dx\right)^{\frac{2}{p+1}}dt$$

$$\leq C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt \ . \quad (3.12)$$

By an analogous estimates of (3.11) and (3.12), we obtain the following estimate,

$$\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \rho_2\psi_t^2 dxdt \leq C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt \ , \quad (3.13)$$

$$\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \rho_1 w_t^2 dxdt \leq C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt. \quad (3.14)$$

Substituting the estimates (3.6)-(3.11) into the right-hand side of (3.4), we obtain that

$$\int_S^T E(t)^{\frac{p+1}{2}}dt \leq CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)dt+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt$$

$$\leq CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt. \quad (3.15)$$

Using the Young inequality, for any fixed $\varepsilon>0$ we have

$$\int_S^T E(t)^{\frac{p+1}{2}}dt \leq CE(S)^{\frac{p+1}{2}}+C\int_S^T E(t)^{\frac{p-1}{2}}\left(-E'(t)\right)^{\frac{2}{p+1}}dt$$

$$\leq CE(S)^{\frac{p+1}{2}}+C\varepsilon\int_S^T E(t)^{\frac{p+1}{2}}dt+C\varepsilon^{\frac{1-p}{2}}\int_S^T\left(-E'(t)\right)dt. \quad (3.16)$$

Therefore

$$(1-C\varepsilon)\int_S^T E(t)^{\frac{p+1}{2}}dt$$

$$\leq CE(S)^{\frac{p+1}{2}} + C\varepsilon^{\frac{1-P}{2}}\int_S^T(-E'(t))dt \leq C\left(1+\varepsilon^{\frac{1-p}{2}}\right)\left(1+E(S)^{\frac{p-1}{2}}\right)E(S). \tag{3.17}$$

Choosing $0<\varepsilon<1$ such that $1-C\varepsilon>0$ and using the non increasingness of the energy, we have, for all $0\leq S<T<+\infty$,

$$\int_S^T E(t)^{\frac{p+1}{2}}dt \leq c\left(1+E(0)^{\frac{p-1}{2}}\right)E(S).$$

Letting $T\to+\infty$, this yields the following estimate:

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}}dt \leq CE(S). \tag{3.18}$$

The constant C in (3.18) is independent of S and $E(0)$, then, by Lemma 2.1, we get the result. If $p=1$, we easily obtain from (3.9) and (3.12)

$$\left|-\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \alpha_1(x)\varphi g_1(\varphi_t)dxdt\right| \leq CE(S)^{p+1},$$

$$\int_S^T E(t)^{\frac{p-1}{2}}\int_0^L \rho_1\varphi_t^2 dxdt \leq CE(S)^{p+1},$$

then, by an analogous process of the above and by Lemma 2.1, we get the result.

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Competing Interests

Authors have declared that no competing interests exist.

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