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$*^s$-tuple and $*^n$-tuple of Covariant Functors

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Abstract

A right A-module M is a $*^s$-module provided that M is self-small and any exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow Q \longrightarrow 0,$$

with $L, Q \in$ Stat(M) remains exact after applying the functor $Hom_A(M, -)$ if and only if $N \in$ Stat(M). A right A-module M is called a $*^n$-module if it is self-small, $(n+1)$-quasi-projective and n-$Pres(M) = (n+1)$-$Pres(M)$. In this work we generalize the concepts of $*^s$-module and $*^n$-modules to the concepts of $*^s$-tuple and $*^n$-tuple of Contravariant Functors between abelian categories.

Keywords: $*^s$-module, $*^n$-modules, contravariant functor, right adjoint functors.

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1 Introduction

In [1], Wei introduced the concept of $*^s$-modules. A right A-module M is a $*^s$-module provided that M is self-small and any exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow Q \longrightarrow 0,$$

with $L, Q \in$ Stat(M) remains exact after applying the functor $Hom_A(M, -)$ if and only if $N \in$ Stat(M), where Stat(M) is the category of all M-static modules.

Following Wei et al, in [2], we call a right A-module M a $*^n$-module if it is self-small, $(n+1)$-quasi-projective and n-$Pres(M) = (n+1)$-$Pres(M)$. Note that a right A-module M is called n-quasi-projective if for any exact sequence

$$0 \longrightarrow N \longrightarrow M^{(I)} \longrightarrow L \longrightarrow 0,$$

where $L \in (n-1)$-$Pres(M)$, the sequence,

$$0 \longrightarrow H_M(X) \longrightarrow H_M(M^{(I)}) \longrightarrow H_M(Y) \longrightarrow 0,$$

is exact, where $H_M = Hom_A(M, -)$.

In this work we generalizes the notion of $*^s$-module and $*^n$-module to $*^s$-tuple and $*^n$-tuple, respectively by generalizing the work in [1] and [2]. We use the same technique of proofs of that papers.

There are many generalizations, in the direction of abstract categories, of many aspects of such theories . Here we give few examples. In [3] Castaño Iglesias et al. consider the equivalences induced by any adjoint pair of covariant functors between complete and cocomplete abelian categories, generalizing the situation of equivalence that induced by the adjoint pair of functors $M \otimes_D -$ and $Hom_A(M, -)$ between the categories of M-static A-modules and M-costatic D-modules, for any left A-module M with endomorphism ring D. $*$-objects, tilting objects, quasi-progenerators and progenerators are such a generalizations by Colpi in [4]. In [5] Happel, Reiten and Smalø have studied aspects of tilting theory for locally finite abelian categories over a commutative artinian ring. On the other hand, In [6] Castaño-Iglesias generalizes the notion of costar module to Grothendieck categories. Pop in [7] generalizes the notion of finitistic n-self-cotilting module to finitistic n-F-cotilting object in abelian categories and he describes a family of dualities between some special abelian categories. Breaz and Pop in [8] generalize a duality exhibited in [9, Theorem 2.8] to abelian categories. In [10], the author generalizes the notion of r-costar module to r-costar pair of contravariant functors between abelian categories, by generalizing the work in [11]. In [12] the author generalize the work in [13] by generalizing the notion of Co-$*^n$-modules to a Co-$*^n$-tuple of contravariant functors between abelian categories.

2 Preliminaries

Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ be an additive covariant functor which has a left adjoint functor $G : \mathfrak{D} \longrightarrow \mathfrak{C}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are two abelian categories. Then there are two natural transformations $\delta : GF \longrightarrow 1_{\mathfrak{C}}$ and $\rho : 1_{\mathfrak{D}} \longrightarrow FG$. Moreover the following identities are satisfied for each $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$.

$$F(\delta_X) \circ \rho_{F(X)} = 1_{F(X)} \text{ and } G(\rho_Y) \circ \delta_{G(Y)} = 1_{G(Y)}.$$

Note that F is left exact and G is right exact, since they are adjoint on the left. The pair (F, G) is called an equivalence if there are functorial isomorphisms $GF \simeq 1_{\mathfrak{C}}$ and $FG \simeq 1_{\mathfrak{D}}$. An object X of $\mathfrak{C}$ (respectively Y of $\mathfrak{D}$) is called F-static (respectively, F-costatic) in case δ_X (respectively, ρ_Y) is an isomorphism. By Stat(F) we will denote the full subcategory of all F-static objects. As well by

Costat(F) we will denote the full subcategory of all G-costatic objects. It is clear that the functors F and G induce an equivalence between the categories Stat(F) and Costat(F).

Let U be an object in $\mathfrak{C}$. For an object X in an abelian category $\mathfrak{C}$, we say that X is U-generated if there is an exact sequence

$$U^{(I)} \longrightarrow X \longrightarrow 0,$$

where I is an index set. If there is an exact sequence

$$U^{(I_2)} \longrightarrow U^{(I_1)} \longrightarrow X \longrightarrow 0,$$

where each I_i is an index set, then X is said to be U-presented. We say that X is n-U-presented if there is an exact sequence

$$U^{(I_{n-1})} \longrightarrow U^{(I_{n-2})} \longrightarrow ... \longrightarrow U^{(I_1)} \longrightarrow U^{(I_0)} \longrightarrow X \longrightarrow 0,$$

where each I_i is an index set and n is a positive integer. We denote by $Gen(U)$, $Pres(U)$ and n-$Pres(U)$ the classes of all U-generated, U-presented and n-U-presented objects respectively. It is clear that $(n+1)$-$Pres(U) \subseteq n$-$Pres(U)$, for every positive integer n.

An object U in $\mathfrak{C}$ is called F-small if for any set I, there is a canonical isomorphism $F(U^{(I)}) \cong F(U)^{(I)}$. The object U is called n-F-quasi-projective if for any exact sequence

$$0 \longrightarrow X \longrightarrow U^{(I)} \longrightarrow Y \longrightarrow 0,$$

where $Y \in (n-1)$-$Pres(U)$, the sequence,

$$0 \longrightarrow F(X) \longrightarrow F(U^{(I)}) \longrightarrow F(Y) \longrightarrow 0,$$

is exact.

Let $V \in \mathfrak{D}$ be a projective object in $\mathfrak{D}$ and let $U = G(V)$. If U is F-static, the tuple (F, G, V, U) is called a $*^n$-tuple, where n is a positive integer, if:
(i) U is F-small,
(ii) $(n+1)$-F-quasi-projective,
(iii) n-$Pres(U) = (n+1)$-$Pres(U)$.

Let $V \in \mathfrak{D}$ be a projective object in $\mathfrak{D}$ and let $U = G(V)$. If U is F-static, we say that the tuple (F, G, V, U) is a $*^s$-tuple provided that U is F-small and any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

with $Y, Z \in$ Stat(F) remains exact after applying the functor F if and only if $Q \in$ Stat(F).

From now on we suppose that $\mathfrak{D}$ has enough projectives i.e. for every object $X \in \mathfrak{D}$ there is a projective object $P \in \mathfrak{D}$ and an epimorphism $P \longrightarrow X \longrightarrow 0$. It is clear that we can construct a projective resolution for any object X. Suppose we have a projective resolution of X

$$P: \quad ... \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

This gives rise to the sequence

$$0 \longrightarrow G(X) \longrightarrow G(P_0) \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow ...,$$

and the cochain complex $G(P)$, which we can compute its cohomology at the n-th spot (the kernel of the map from $G(P_n)$ modulo the image of the map to $G(P_n)$) and denote it by $H^n(G(P))$. We define $R^nG(X) = H^n(G(P))$ as the n-th right derived functor of G. For the functor G we define ${}^{\perp}T_G^{i \geqslant n} = \{X \in \mathfrak{D} : R^iG(X) = 0$ for every $i \geqslant n\}$.

We will denote by proj($\mathfrak{D}$) the full subcategory of all projective objects in $\mathfrak{D}$.

Lemma 2.1. *[3, Lemma 1.4]Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be a pair of covariant functors and $U \in \mathfrak{C}$. If $U^{(I)}$ if F-static for every set I then δ_X is an epimorphism for every $X \in Gen(U)$.*

Lemma 2.2. *Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be a pair of covariant functors. Let V be a projective generator in $\mathfrak{D}$ and let $U = G(V)$. For any $Y \in \mathfrak{D}$, if $R^iG(Y) = 0$ for $1 \le i \le n$, then $G(Y) \in (n + 2)$-$Pres(U)$.*

Now we will prove something dual to [6, Lemma 2.2].

Lemma 2.3. *Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be a pair of covariant functors. Let X be an object in $\mathfrak{C}$ and V be an F-costatic generator of $\mathfrak{D}$. Let $U = G(V)$ in $\mathfrak{C}$. For every $X \in Gen(U)$, there exists an epimorphism $U^{(I)} \stackrel{f}{\longrightarrow} X \longrightarrow 0$, such that $F(f)$ is an epimorphism.*

Proof. Since $F(X)$ is generated by V in $\mathfrak{D}$, there is an epimorphism $V^{(I)} \stackrel{h}{\longrightarrow} F(X) \longrightarrow 0$. Applying the functor G to this epimorphism we get the epimorphism

$$G(V^{(I)}) = U^{(I)} \stackrel{G(h)}{\longrightarrow} GF(X) \longrightarrow 0.$$

The composition $f = \delta_X \circ G(h)$ provides the requested epimorphism, since δ_X is an epimorphism by Lemma 2.1. Now we have the following commutative square

$$\begin{array}{cccc} V^{(I)} & \stackrel{h}{\longrightarrow} & F(X) & \\ \downarrow\rho_{V^{(I)}} & \nearrow_{F(f)} & F(\delta_X)\uparrow\downarrow\rho_{F(X)} & \\ FG(V^{(I)}) & \longrightarrow & FGF(X) & \longrightarrow \quad 0 \end{array} .$$

Since $F(\delta_X) \circ \rho_{F(X)} = 1_{F(X)}$, then $F(f) \circ \rho_{V^{(I)}} = h$. Since $\rho_{V^{(I)}}$ is an epimorphism, $F(f)$ is an epimorphism. □

3 $*^s$-tuple of Covariant Functors

In this section we suppose that we have $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ as an additive covariant functor which has a left adjoint functor $G : \mathfrak{D} \longrightarrow \mathfrak{C}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are two abelian categories. As well we suppose that V is an F-costatic projective generator in $\mathfrak{D}$ and $U = G(V)$.

Proposition 3.1. *Suppose that the functor F respects the exactness of any sequence in the form*

$$0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0.$$

Suppose that U is F-small. For any $X \in$ Stat(F), there is an infinite exact sequence

$$... \longrightarrow U^{(I_n)} \longrightarrow ... \longrightarrow U^{(I_1)} \longrightarrow X \longrightarrow 0$$

which remains exact after applying the functor F.

Proof. Let $X \in$ Stat(F). Then $F(X) \in$ Costat(F), so by assumption there is an exact sequence

$$V^{(I)} \longrightarrow F(X) \longrightarrow 0.$$

Applying the functor G we have an exact sequence

$$0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0,$$

for some $Y \in \mathfrak{C}$. Since (F, G, U, V) is a $*^s$-tuple, the last sequence is exact after applying the functor F, that is we have an exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^{(I)}) \longrightarrow F(X) \longrightarrow 0.$$

Applying the functor G again we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & GF(Y) & \longrightarrow & GF(U^{(I)}) & \longrightarrow & GF(X) & & \\ & & \downarrow\delta_X & & \downarrow\delta_{U^{(I)}} & & \downarrow\delta_Y & & . \\ 0 & \longrightarrow & Y & \longrightarrow & U^{(I)} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Since $X, U^{(I)} \in$ Stat(F), $Y \in$ Stat(F), by snake lemma. By repeating the process to Y, and so on, we finally obtain the desired exact sequence. ☐

From now on in this section we will assume that the functor F respects the exactness of the sequences of the form

$$0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0.$$

Proposition 3.2. *Let (F, G, U, V) be a $*^s$-tuple, then Costat(F) $\subseteq^\perp T_G^{i\geqslant 1}$.*

Proof. Let $X \in$ Costat(F), then $G(X) \in$ Stat(F) and hence by Proposition 3.1, there is an infinite exact sequence

$$... \longrightarrow U^{(I_n)} \longrightarrow ... \longrightarrow U^{(I_1)} \longrightarrow G(X) \longrightarrow 0 \tag{3.1}$$

which remains exact after applying the functor F. So we have an exact sequence

$$... \longrightarrow F(U^{(I_n)}) \longrightarrow ... \longrightarrow F(U^{(I_1)}) \longrightarrow FG(X) \longrightarrow 0$$

Again the last sequence remains exact after applying the functor G, since we get a sequence isomorphic to sequence (3.1), because $G(X), U^{(I_i)}$, for each i, are F-static. We obtain that Costat(F) $\subseteq^\perp T_G^{i\geqslant 1}$by dimension shifting. ☐

Proposition 3.3. *If Costat(F) $\subseteq^\perp T_G^{i\geqslant 1}$and $^\perp T_G^{i\geqslant 0} = 0$ and U is F-small, then (F, G, U, V) is a $*^s$-tuple.*

Proof. Let

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0 \tag{3.2}$$

be an exact sequence with $Y, Z \in$ Stat(F). Assume that we have the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0,$$

after applying the functor F. Applying the functor G, we get an exact sequence

$$L^1G(F(Z)) = 0 \longrightarrow GF(X) \longrightarrow GF(Y) \longrightarrow GF(Z) \longrightarrow 0,$$

since $F(Z) \in$ Costat(F)$\subseteq^\perp T_G^{i\geqslant 1}$. Hence we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & GF(X) & \longrightarrow & GF(Y) & \longrightarrow & GF(Z) & \longrightarrow & 0 \\ & & \downarrow\delta_X & & \downarrow\delta_Y & & \downarrow\delta_Z & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

Since $Y, Z \in$ Stat(F), δ_Y and δ_Z are isomorphisms. Now it is clear that δ_X is an isomorphism which means that $X \in$ Stat(F). Conversely, suppose that $X \in$ Stat(F). Applying the functor F to the sequence (3.2), we get an exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow Q \longrightarrow 0, \tag{3.3}$$

where $Q = \mathrm{Im}(F(g))$. Hence we can get the exact sequence

$$0 \longrightarrow Q \xrightarrow{\mathrm{i}} F(Z) \longrightarrow W \longrightarrow 0, \tag{3.4}$$

for some $W \in \mathfrak{D}$, and $\mathfrak{i}$ is the inclusion map . Applying the functor G to the sequence (3.3), we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow L^1G(Q) & \longrightarrow & GF(X) & \longrightarrow & GF(Y) & \longrightarrow & G(Q) & \longrightarrow & 0 \\ & & \downarrow \delta_X & & \downarrow \delta_Y & & \downarrow \alpha & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array} \quad (3.5)$$

where $\alpha = \delta_Z \circ G(\mathfrak{i})$. Note that δ_X and δ_Y are isomorphisms, since $X, Y \in$ Stat(F). It is clear from the diagram that $L^1G(Q) = 0$. Now $L^iG(Q) = 0$ for all $i \geqslant 2$, by dimension shifting, since $F(X), F(Y) \in$ Costat(F)$\subseteq^{\perp} T_G^{i\geqslant 1}$. Hence $Q \in^{\perp} T_G^{i\geqslant 1}$. Now applying the functor G to sequence (3.4), we get the long exact sequence

$$0 \longrightarrow L^1G(W) \longrightarrow G(Q) \overset{G(\mathfrak{i})}{\longrightarrow} GF(Z) \longrightarrow G(W) \longrightarrow 0, \quad (3.6)$$

since $Q \in^{\perp} T_G^{i\geqslant 1}$and $F(Z) \in$ Costat(F)$\subseteq^{\perp} T_G^{i\geqslant 1}$. Hence by dimension shifting $W \in^{\perp} T_G^{i\geqslant 2}$. Note that $\alpha = \delta_Z \circ G(\mathfrak{i})$ in diagram (3.5) is an isomorphism, since δ_X and δ_Y are isomorphisms. Hence $G(\mathfrak{i})$ is an isomorphism, since δ_Z is an isomorphism, so from sequence (3.6), $R^1G(W) = 0 = G(W)$. We conclude that $W \in^{\perp} T_G^{i\geqslant 0}$. Since ${}^{\perp}T_G^{i\geqslant 0} = 0$ by assumptions, $W = 0$ and hence from sequence (3.4) $Q \cong F(Z)$ canonically. Therefore the functor F preserves the exactness of the exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in Stat(F). We conclude that the pair (F, G, U, V) is a $*^s$-tuple. □

Suppose we have the following exact sequence in $\mathfrak{D}$

$$0 \longrightarrow X \longrightarrow P_2 \longrightarrow P_1 \longrightarrow Y \longrightarrow 0,$$

where P_2, P_1 are projective objects in $\mathfrak{D}$ and $Y \in^{\perp} T_G^{i\geqslant 1}$. Applying the functor G we get the following exact sequence

$$L^1G(Y) = 0 \longrightarrow G(X) \longrightarrow G(P_2) \longrightarrow G(P_1) \longrightarrow G(Y) \longrightarrow 0.$$

Applying the functor F we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & P_2 & \longrightarrow & P_1 \\ & & \downarrow \rho_X & & \downarrow \rho_{P_2} & & \downarrow \rho_{P_1} \\ 0 & \longrightarrow & FG(X) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1) \end{array} .$$

If proj($\mathfrak{D}$) $\subseteq$ Costat(F), then it is clear that $X \in$ Costat(F).

Proposition 3.4. *Let (F, G, U, V) be a $*^s$-tuple and suppose that proj($\mathfrak{D}$) $\subseteq$ Costa(F). Then* ${}^{\perp}T_G^{i\geqslant 0} = 0$.

Proof. For any $Y \in^{\perp} T_G^{i\geqslant 0}$, we can build the following exact sequence in $\mathfrak{D}$

$$0 \longrightarrow X \longrightarrow P_2 \longrightarrow P_1 \longrightarrow Y \longrightarrow 0,$$

where P_2, P_1 are projective objects and X an object in $\mathfrak{D}$. By the argument before the proposition it is clear that $X \in$ Costa(F) and hence $G(X) \in$ Stat(F). Applying the functor G we get the following exact sequence

$$L^1G(Y) = 0 \longrightarrow G(X) \longrightarrow G(P_2) \longrightarrow G(P_1) \longrightarrow 0.$$

Since (F, G, U, V) is a $*^s$-tuple, applying the functor F we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & Y \\ & & \downarrow \rho_X & & \downarrow \rho_{P_2} & & \downarrow \rho_{P_1} & & \\ 0 & \longrightarrow & FG(X) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1) & \longrightarrow & 0 \end{array} .$$

Thus it is clear that $Y \cong 0$. □

Now we are able to give the following characterization of $*^s$-tuple.

Theorem 3.1. *Let proj($\mathfrak{D}$) $\subseteq$ Costat(F). Then (F,G,U,V) is a $*^s$-tuple if and only if Costat(F) $\subseteq {}^{\perp}T_G^{i\geqslant 1}$ and ${}^{\perp}T_G^{i\geqslant 0}=0$.*

Proof. By Propositions 3.2, 3.3 and 3.4. □

Corollary 3.2. *If proj($\mathfrak{D}$) $\subseteq$Costat(G), then the following are equivalent.*
(1) (F,G,U,V) is a $*^s$-tuple.
(2) For any exact sequence

$$0\longrightarrow X\longrightarrow U^{(I)}\longrightarrow Y\longrightarrow 0,$$

with $Y\in$Stat(F), then $X\in$Stat(F) if and only if the exact sequence remains exact after applying the functor F.

Proof. (1) $\Longrightarrow$ (2) follows from the definition of $*^s$-tuple.
(2) $\Longrightarrow$ (1) the proof goes the same as the proofs of Propositions 3.1, 3.2, 3.4 and Theorem 3.1. □

4 $*^n$-tuple of Covariant Functors

In this section we suppose that we have $F:\mathfrak{C}\longrightarrow\mathfrak{D}$ as an additive covariant functor which has a left adjoint functor $G:\mathfrak{D}\longrightarrow\mathfrak{C}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are two abelian categories. As well we suppose that V is an F-costatic projective generator in $\mathfrak{D}$ and $U=G(V)$.

Proposition 4.1. *Suppose that (F,G,V,U) is a $*^n$-tuple. Then for any $X\in n\text{-}Pres(U)$, δ_x is an isomorphism and $L^iG(F(X))=0$, for every $i\geq 1$.*

Proof. Let $X\in n\text{-}Pres(U)$. It follows that $X\in(n+1)\text{-}Pres(U)$, by assumptions. Hence there is an exact sequence

$$0\longrightarrow Y\longrightarrow U^{(I)}\longrightarrow X\longrightarrow 0,$$

where $Y\in n\text{-}Pres(U)$. Since (F,G,V,U) is a $*^n$-tuple we have the exact sequence

$$0\longrightarrow F(Y)\longrightarrow F(U^{(I)})\longrightarrow F(X)\longrightarrow 0,$$

after applying the functor F. Applying the functor G to the last sequence we get an exact sequence

$$L^1G(F(X)\longrightarrow GF(Y)\longrightarrow GF(U^{(I)})\longrightarrow GF(X)\longrightarrow 0,$$

and the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^1G(F(X)) & \longrightarrow & GF(Y) & \longrightarrow & GF(U^{(I)}) & \longrightarrow & GF(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\delta_Y & & \downarrow\delta_{U^{(I)}} & & \downarrow\delta_X & & \\ & & 0 & \longrightarrow & Y & \longrightarrow & U^{(I)} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

By Lemma 2.1, δ_Y is a epimorphism. By Snake Lemma, it follows that δ_X is an isomorphism since $\delta_{U^{(I)}}$ is an isomorphism. Then δ_Y is also an isomorphism by a similar argument. Hence, $L^1G(F(X))=0$, by commutativity of the left square. Since $Y\in n\text{-}Pres(U)$, $L^1G(F(Y)=0$. Then we can get the assertion inductively. □

Theorem 4.1. *The following conditions are equivalent*
(1) (F,G,V,U) is a $*^n$-tuple,
(2) i) U is F-small
(ii) For any exact sequence $0\longrightarrow Y\longrightarrow U^{(I)}\longrightarrow X\longrightarrow 0$, where $X\in n\text{-}Pres(U)$ and I a set, it remains exact after applying the functor F if and only if $Y\in n\text{-}Pres(U)$.

Proof. (1) ⇒ (2) Suppose that we have an exact sequence $0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0$, where $X \in n\text{-}Pres(U)$ and I a set. Assume that $Y \in n\text{-}Pres(U)$. Since (F, G, V, U) is a $*^n$-tuple, we get the exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^{(I)}) \longrightarrow F(X) \longrightarrow 0.$$

Conversely, assume that the sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^{(I)}) \longrightarrow F(X) \longrightarrow 0$$

is exact. Applying the functor G we get the following long exact sequence

$$\begin{gathered} .... \longrightarrow L^1G(F(Y)) \longrightarrow L^1G(F(U^{(I)})) \longrightarrow L^1G(F(X)) \longrightarrow GF(Y) \longrightarrow \\ GF(U^{(I)}) \longrightarrow GF(X) \longrightarrow 0 \end{gathered} \tag{4.1}$$

By Proposition 4.1, δ_X is an isomorphism and $L^iG(F(X)) = 0$ for any $i \geq 1$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & GF(Y) & \longrightarrow & GF(U^{(I)}) & \longrightarrow & GF(X) & \longrightarrow & 0 \\ & & \downarrow_{\delta_Y} & & \downarrow_{\delta_{U^{(I)}}} & & \downarrow_{\delta_X} & & \\ 0 & \longrightarrow & Y & \longrightarrow & U^{(I)} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

It is clear, by Snake Lemma, that δ_Y is an isomorphism. From the exactness of sequence (4.1) we conclude that $L^iG(F(Y)) \cong L^iG(F(U^{(I)})) = 0$ for any $i \geq 1$, so by Lemma 2.2, $Y \cong GF(Y) \in n\text{-}Pres(U)$. For any $X \in n\text{-}Pres(U)$, $X \in (n+1)\text{-}Pres(U)$, by definition. So we have an exact sequence $0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0$, with $Y \in n\text{-}Pres(U)$.
(2)⇒ (1) It is enough to prove $n\text{-}Pres(U) = (n+1)\text{-}Pres(U)$. If $X \in n\text{-}Pres(U)$, then $F(X)$ is V-generated over $\mathfrak{D}$, thus by Lemma 2.3, there exists an exact sequence $0 \longrightarrow Y \longrightarrow U^{(I)} \longrightarrow X \longrightarrow 0$, which remains exact after applying the functor F. Then $Y \in n\text{-}Pres(U)$, hence $X \in (n+1)\text{-}Pres(U)$. □

Proposition 4.2. *Let (F, G, V, U) be a $*^n$-tuple. Then G is an exact functor in $F(n\text{-}Pres(U))$. Moreover $F(n\text{-}Pres(U)) = ^{\perp} T_G^{i\geqslant 1}$.*

Proof. By Proposition 4.1 we have $F(n\text{-}Pres(U)) \subseteq^{\perp} T_G^{i\geqslant 1}$ and G is an exact functor in $F(n\text{-}Pres(U))$. Conversely, for any $X \in^{\perp} T_G^{i\geqslant 1}$, by Lemma 2.2, $G(X) \in n\text{-}Pres(U)$. Since V is a generator in $\mathfrak{D}$, there is an exact sequence $0 \longrightarrow Y \longrightarrow V^{(I)} \longrightarrow X \longrightarrow 0$, where I is a set. If we apply the functor G we get the long exact sequence

$$\begin{gathered} ... \longrightarrow L^1G(Y) \longrightarrow L^1G(V^{(I)}) \longrightarrow L^1(X) \longrightarrow G(Y) \longrightarrow \\ G(V^{(I)}) \longrightarrow G(X) \longrightarrow 0 \end{gathered}$$

By assumption $L^iG((X)) = 0$ for any $i \geq 1$. Since $L^iG((V^{(I)})) = 0$, for any $i \geq 1$, $L^iG((Y)) = 0$ for any $i \geq 1$, by the exactness. Thus $Y \in^{\perp} T_G^{i\geqslant 1}$ and hence by Lemma 2.2, $G(Y) \in n\text{-}Pres(U)$. Since (F, G, V, U) is a $*^n$-tuple, applying the functor F to the following sequence

$$0 \longrightarrow G(Y) \longrightarrow G(V^{(I)}) \longrightarrow G(X) \longrightarrow 0$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & V^{(I)} & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow_{\rho_Y} & & \downarrow_{\rho_{V^{(I)}}} & & \downarrow_{\rho_X} & & \\ 0 & \longrightarrow & FG(Y) & \longrightarrow & FG(V^{(I)}) & \longrightarrow & FG(X) & \longrightarrow & 0 \end{array}$$

Hence by Snake Lemma, ρ_X is an epimorphism, since $\rho_{V^{(I)}}$ is an isomorphism. Similarly ρ_Y is also an epimorphism. Thus, ρ_X is an isomorphism and therefore $X \cong FG(X) \in F(n\text{-}Pres(U))$. So $F(n\text{-}Pres(U)) = ^{\perp} T_G^{i\geqslant 1}$. □

Proposition 4.3. *Let (F, G, V, U) be a $*^n$-tuple. Then F preserves any exact sequence in n-$Pres(U)$.*

Proof. Let $0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0$ be an exact sequence in n-$Pres(U)$. Applying the functor F we get the following long exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\alpha} R^1F((X)) \longrightarrow ....$$

Thus we can get the following two exact sequences

$$0 \longrightarrow W \longrightarrow F(Z) \longrightarrow Q \longrightarrow 0,$$

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow W \longrightarrow 0,$$

where $Q = \operatorname{Im}\alpha$ and $W = \operatorname{Im} F(G)$. Applying the functor G to the last sequence we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} L^1G(W) & \longrightarrow & GF(X) & \longrightarrow & GF(Y) & \longrightarrow & G(W) & \longrightarrow & 0 \\ \downarrow & & \downarrow_{\delta_X} & & \downarrow_{\delta_Y} & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}.$$

It is clear by Proposition 4.1 that δ_X and δ_Y are isomorphisms and $L^iG(F(X)) = 0 = L^iG(F(Y))$, for any $i \geq 1$. By Snake Lemma, $Z \cong G(W)$ and by the exactness $L^iG(W) = 0$ for any $i \geq 1$. Hence by Proposition 4.2, $W = F(D)$ for some $D \in n$-$Pres(U)$. Therefore

$$W = F(D) \cong F(GF(D)) = FG(F(D)) = FG(W) \cong F(Z).$$

Hence $Q = 0$. □

Theorem 4.2. *The following conditions are equivalent:*
(1) (F, G, V, U) is a $*^n$-tuple.
(2) (i) U is F-small;
(ii) For any exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ whit $Y, Z \in n$-$Pres(U)$, we have $X \in n$-$Pres(U)$ if and only if $0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$ is exact.

Proof. (1)⇒(2) The necessity follows from Proposition 4.3 and the sufficiency from a similar proof to that of (1)⇒(2) in Theorem 4.1.
(2)⇒(1) It follows from (2)⇒(1) in Theorem 4.1. □

Proposition 4.4. *Let (F, G, V, U) be a $*^n$-tuple. Then n-$Pres(U)$ is closed under extensions if and only if n-$Pres(U) \subseteq^{\perp} T_F^1 = \{X \in \mathfrak{C} : R^1F(X) = 0\}$.*

Proof. Suppose that n-$Pres(U)$ is closed under extensions. For any $X \in n$-$Pres(U)$ one can construct an exact sequence using the canonical maps to get an extension $0 \longrightarrow X \longrightarrow Y \longrightarrow U \longrightarrow 0$ of X by U. We have $Y \in n$-$Pres(U)$, by assumption. By Proposition 4.3, F preserves any exact sequence in n-$Pres(U)$, so applying F to the last exact sequence we get the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(U) \longrightarrow 0,$$

thus by the exactness, $R^1F(X) = 0$, so $X \in^{\perp} T_F^1$ and hence n-$Pres(U) \subseteq^{\perp} T_F^1$. Conversely. For any extension $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, of X by Z, where $X, Z \in n$-$Pres(U)$, the induced sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0,$$

is exact by assumption. According to Proposition 4.1, both δ_X and δ_Z are isomorphisms and $F(X), F(Z) \in^{\perp} T_G^{i\geq 1}$. Then it is clear that δ_Y is an isomorphism and $F(Y) \in^{\perp} T_G^{i\geq 1}$. Hence by Lemma 2.2, we have $Y \cong GF(Y) \in n$-$Pres(U)$. □

Theorem 4.3. *The following conditions are equivalent:*
(1) (F, G, V, U) *is a* $*^n$*-tuple.*
(2) There is an equivalence.

$$G :^{\perp} T_G^{i\geq 1} \rightleftarrows n\text{-}Pres(U) : F$$

Proof. (1)⇒(2) By Propositions 4.1 and Propositions 4.2.
(2)⇒(1) Since $V^{(I)} \in^{\perp} T_G^{i\geq 1}$, we get $F(U^{(I)}) \cong F(G(V)^{(I)}) \cong FG(V^{(I)}) \cong V^{(I)} \cong F(U)^{(I)}$. So U is F-small. For any $X \in n\text{-}Pres(U)$, by assumption $G(F(X)) \cong X$ and $F(X) \in^{\perp} T_G^{i\geq 1}$, thus $G(F(X)) \cong X \in (n+1)\text{-}Pres(U)$, by Lemma 2.2. So $n\text{-}Pres(U) = (n+1)\text{-}Pres(U)$. Now let

$$0 \longrightarrow X \longrightarrow U^{(I)} \longrightarrow Y \longrightarrow 0$$

be an exact sequence, with $X \in n\text{-}Pres(U)$ and I a set. We can get the following exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(U^{(I)}) \longrightarrow F(Y) \longrightarrow Q \longrightarrow 0,$$

where $Q = \operatorname{Im}\alpha$, where $\alpha : F(Y) \longrightarrow R^1F(X)$. By using argument similar to that in Proposition 4.3, we conclude that $Q = 0$, which means that we have an exact sequence $0 \longrightarrow F(X) \longrightarrow F(U^{(I)}) \longrightarrow F(Y) \longrightarrow 0$. Thus (F, G, V, U) is a $*^n$-tuple. □

Proposition 4.5. *Let* U *be a* F*-small . Assume that* $n\text{-}Pres(U) =^{\perp} T_F^{i\geq 1}$*. Then* (F, G, V, U) *is a* $*^n$*-tuple.*

Proof. Let

$$0 \longrightarrow X \longrightarrow U^{(I)} \longrightarrow Y \longrightarrow 0,$$

be an exact sequence with $Y \in n\text{-}Pres(U)$ and I a set. We can get the following long exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(U^{(I)}) \longrightarrow F(Y) \longrightarrow R^1F(X) \longrightarrow$$
$$R^1F(U^{(I)}) \longrightarrow R^1F(Y) \longrightarrow ....$$

Note that $Y, U^{(I)} \in n\text{-}Pres(U) =^{\perp} T_F^{i\geq 1}$, so by exactness, $R^iF(X) = 0$, for every $i \geq 2$. Now $R^1F(X) = 0$ if and only if $X \in {}^{\perp}T_F^{i\geq 1} = n\text{-}Pres(U)$. So by Theorem 4.1 we get the desired result. □

5 Conclusion

In this work we generalize the concepts of $*^s$-module and $*^n$-modules to the concepts of $*^s$-tuple and $*^n$-tuple of Contravariant Functors between abelian categories.

Competing Interests

The author declares that no competing interests exist.

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