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On Alexandroff Shadow Spaces

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Abstract

Each Alexandroff space X has a corresponding shadow space $[X]$ which is T_0 Alexandroff space. In this paper, we study Alexandroff spaces and their properties via their shadow spaces. The definitions and the concepts such as Artinian, Noetherian, maximal points and minimal points, that are defined on T_0 Alexandroff space carry over to any Alexandroff space. We prove that an Alexandroff space X is connected (compact) iff its shadow space $[X]$ is connected (compact). Moreover, X need not be scattered or α-scattered. We give a study of preopen, semi-open, and α-open sets on X.

Keywords: Alexandroff spaces; generalized open sets; $\alpha-$open sets; preopen sets; shadow spaces.
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1 Introduction

An *Alexandroff space* (briefly $A-$space) (or *minimal neighborhood space*) X is a topological space in which the arbitrary intersection of open sets is open. These spaces were first introduced by P.

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Alexandroff in 1937 [1]. In these spaces, each element x has a minimum open neighborhood set $V(x)$ which is the intersection of all open sets containing x. For every T_o $A-$space (X,τ), there is a corresponding poset $(X,\leq_\tau)$, in one to one and onto way, where each one of them is completely determined by the other. If $(X,\leq)$ is a poset, then $\mathcal{B}=\{\uparrow x : x\in X\}$ forms a base for a topology on X denoted by $\tau_\leq$ which is a T_0 $A-$space. Moreover, if (X,τ) is an A-topological space, we define the pre-order $\leq_\tau$, called (*Alexandroff*) *specialization pre-order*, by: $a\leq_\tau b$ if and only if $a\in\overline{\{b\}}$. This specialization pre-order is partial order if and only if (X,τ) is T_o . On the other hand, if (X,τ) is a T_o $A-$space and if $\leq_\tau$ is its specialization order, then the induced topology by the specialization order is the original topology. Also if $(X,\leq)$ is a poset and $\tau(\leq)$ its induced T_o $A-$topology, then the specialization order $\leq_{(\tau_\leq)}=\leq$. So we consider $(X,\tau(\leq))$ to be a T_o $A-$space (X,τ) together with its specialization order $\leq$. We see that $\forall x\in X$, $V(x)$ equals $\uparrow x$; the up set of x in the corresponding poset. A poset $(X,\leq)$ satisfies the *ascending chain condition* (ACC) , if for any increasing sequence $x_1\leq x_2\leq\cdots\leq x_n\leq\cdots$ in X, there exists $k\in\mathbb{N}$ such that $x_k=x_{k+1}=\cdots$. The dual of (ACC) is the *descending chain condition* (DCC). If a poset satisfies both ACC and DCC, we say that X is of *finite chain condition* (FCC). Given a poset $(X,\leq)$, the set of all maximal elements is denoted by $M(X)$ (or simply by M) and the set of all minimal elements is denoted by $m(X)$ (or simply by m). Moreover, for each $x\in X$, we define $\hat{x}$ to be the set of all maximal elements grater than or equal to x and $\check{x}$ to be the set of all minimal elements less than or equal to x. That is, $\hat{x}=\uparrow x\cap M$ and $\check{x}=\downarrow x\cap m$.

A T_o $A-$space whose corresponding poset satisfies the ACC is called *Artinian* T_o $A-$space, and whose corresponding poset satisfies the DCC is called *Noetherian* T_o $A-$space [2]. If X is a topological space and $\mathfrak{D}$ a partition of X, then $\mathfrak{D}$ can be topologized as follows: $\mathfrak{F}\subseteq\mathfrak{D}$ is open in $\mathfrak{D}$ iff $\bigcup_{F\in\mathfrak{F}}F$ is open in X. The topology $\tau_\mathfrak{D}$ on $\mathfrak{D}$ is called the *quotient topology* of X induced by $\mathfrak{D}$, and the open sets U in X where $U=\bigcup\{F\in\mathfrak{F}:\mathfrak{F}\in\tau_\mathfrak{D}\}$ are called *saturated*. It should be noted that not all open sets in X are saturated. Nevertheless, each saturated open set has a corresponding open set in $\mathfrak{D}$. So there is a one to one correspondence between $\tau_\mathfrak{D}$ and the collection of all saturated open sets in X. Given a topological space X. An equivalence relation $\sim$ is defined on X as follows: $x\sim y$ iff x and y cannot be separated by open sets. The set of all equivalence classes $[X]$ forms a partition on X with quotient topology satisfies the separation axiom T_0. Again, not all open sets in X are saturated. In [3], it was proved that if the topology on X is Alexandroff, then each open set in X is saturated with respect to the equivalence relation $\sim$. Hence there is a one to one correspondence between τ on X and the quotient topology on $[X]$. So, the quotient topology on $[X]$ is called the *shadow topology* of τ and denoted by τ_s. The shadow space $([X],\tau_s)$ is a T_0 $A-$space and has a corresponding poset $([X],\leq_s)$. Some of previous studies (see [4] and [2]) used the corresponding posets in proofs of the results on $A-$space that are satisfying the separation axiom T_0. This technique proves to be an easier approach, and we can't use it if X is not T_0. For this end, and to get an extension study including the $A-$spaces that are not satisfying the T_0 axiom, Mahdi in [3], use for a given $A-$space (X,τ) a corresponding shadow space $[X]$ and its induced poset in his study to get some properties of X.

This paper is a continuation study of [3] and the reader should be familiar with the two papers [3] and [2].

2 Preliminary Notes

Let (X,τ) be a topological space and $A\subseteq X$. Then A^c, A', $Int_X(A)$ and $Cl_X(A)$ will denote the complement, the limit points, the interior and the closure of A respectively. For each $A-$space X, we always consider its shadow space $([X],\tau_s)$ with corresponding poset $([X],\leq_s)$. For two classes $[a],[b]$ in $[X]$, we have $[a]\leq[b]$ iff $b\in V(a)$ iff $V(b)\subseteq V(a)$ iff $a\in\overline{b}$. For a subset $A\subseteq X$, we define $[A]=\{[a]:\ a\in A\}$. It may happen that $A\neq B$ but $[A]=[B]$. If A is open in X, and if $x\in A$, then as A saturated, $[x]\subseteq A$. If $x\in X$ with minimal neighbourhood $V(x)$ of x, then $[V(x)]=\uparrow[x]$. (For more information, you can see [3]).

Theorem 2.1. *[2] Let X be an Artinian T_0 $A-$space. Then*

(1) $Int(A) = \emptyset$ *iff* $A \cap M = \emptyset$.

(2) $Cl(A) = \cup\{\downarrow x : x \in M(A)\} = \downarrow M(A)$.

(3) $A' = \cup\{(\downarrow x) \setminus \{x\} : x \in M(A)\} = (\downarrow M(A)) \setminus M(A)$.

(4) *The subset A is dense iff $M \subseteq A$.*

(5) *The subset A is nowhere dense iff $M \cap A = \emptyset$.*

(6) *If $|M| = 1$, then any subset is either dense or nowhere dense.*

Theorem 2.2. *[3] Let X be an $A-$space, $A \subseteq X$, and $x \in X$. Then*

(1) $Cl(x) = \{c : V(x) \subseteq V(c)\}$.

(2) $Cl(A) = \{c : V(x) \subseteq V(c)$ *for some* $x \in A\}$.

Theorem 2.3. *[3] Let X be an $A-$space, $A \subseteq X$. Then A is dense iff $[A]$ is dense.*

Theorem 2.4. *Let X be an $A-$space and A an open (closed) set. Then $a \in A$ iff $[a] \in [A]$.*

Proof. $a \in A$ implies $[a] \in [A]$ follows from the definition of $[A]$. Conversely, if $[a] \in [A]$, then there is $b \in A$ such that $[a] = [b]$. So, $[a] \subseteq A$ and hence $a \in A$.

Theorem 2.5. *If X is an $A-$space and $A \subseteq X$. Then $[Int(A)] \subseteq Int_s([A])$*

Proof. Since $Int(A) \subseteq A$, $[Int(A)] \subseteq [A]$, and since $[Int(A)]$ is open in $[X]$, $[Int(A)] \subseteq Int_s([A])$.

The converse is not always true, as shown in the following example.

Example 2.6. *Let $X = \{1,2,3,4,5\}$, $\tau = \{X, \emptyset, \{4,5\}\}$. Then, $[1] = [2] = [3] = \{1,2,3\}$, $[4] = [5] = \{4,5\}$. So $[X] = \{[1],[4]\}$ and $\tau_s = \{[X], \emptyset, \{[4]\}\}$, which is the Sierpinski topology. Let $A = \{1,4\}$. Then, $Int(A) = \emptyset$ and $[A] = \{[1],[4]\} = [X]$. Therefore $Int_s([A]) = [X]$ and so $Int_s([A])$ is not a subset of $[Int(A)]$.*

Theorem 2.7. *Let X be an $A-$space and let $A \subseteq X$. Then $[Cl(A)] = Cl_s([A])$.*

Proof. Let $[c] \in [Cl(A)]$, then by Theorem 2.4, $c \in Cl(A)$. This implies that there is $x \in A$ such that $V(x) \subseteq V(c)$ and so $[c] \leq [x]$. That is, $[c] \in \downarrow [x] \subseteq Cl_s([A])$. On the other hand, if $[c] \in Cl_s([A])$, then $[c] \in \downarrow [x]$ for some $[x] \in [A]$. Let $a \in A$ be such that $[x] = [a]$. So $[c] \leq [a]$ for some $a \in A$. Then $a \in V(c)$, and hence $[c] \in [Cl(A)]$.

You will notice that for a nonempty set X with an equivalence relation $\sim$ and a corresponding set of equivalence classes $[X] = \{[x] : x \in X\}$, if A, B are two subsets of X, then

(1) $[A \cup B] = [A] \cup [B]$.

(2) $[A \cap B] \subseteq [A] \cap [B]$.

(3) $[A]^c \subseteq [A^c]$.

(4) $A \subseteq B$ implies $[A] \subseteq [B]$.

The reverse inclusions of (2) and (3) and the converse of part (4) are not true in general. To see this, recall Example 2.6, and take $A = \{3,4\}$, $B = \{3,5\}$. Then $[A \cap B] = \{\{1\}\}$, $[A] \cap [B] = [X]$, $[A]^c = \emptyset$ and $[A^c] = [X]$. Fortunately, in an $A-$space the reverse inclusions of (2) and (3) are true if either A or B is open (closed) and the converse of (4) is true if B is open(closed) as shown in the following theorems.

Theorem 2.8. *Let X be an $A-$space , A and B are two subsets of X such that at least one of them is open (closed) set in X. Then*

(1) $[A \cap B] = [A] \cap [B]$.

(2) $[A]^c = [A^c]$

Proof. Follows directly from Theorem 2.4.

Theorem 2.9. *Let X be an $A-$space and let A, B be subsets of X such that B is open (closed). Then $A \subseteq B$ iff $[A] \subseteq [B]$*

Proof. If $x \in A$, then $[x] \in [A]$ and so $[x] \in [B]$. By Theorem 2.4, $x \in B$.

3 Artinian and Noetherian A-spaces

In this section, the definitions and the concepts that are defined on T_0 $A-$space carry over to any $A-$space. Further, we investigate these concepts between an $A-$space and its shadow space.

Definition 3.1. An $A-$space is called *Artinian* if for every points $x_1, x_2, x_3, \cdots$ such that $V(x_1) \supseteq V(x_2) \supseteq V(x_3) \supseteq \cdots$, there is $k \in \mathbb{N}$ such that $V(x_k) = V(x_n)\ \ \forall n \geq k$.

Definition 3.2. An $A-$space is called *Noetherian* if for every points $x_1, x_2, \cdots$ such that $V(x_1) \subseteq V(x_2) \subseteq \cdots$ there is $k \in \mathbb{N}$ such that $V(x_k) = V(x_n)\ \forall n \geq k$.

Theorem 3.1. *An $A-$space X is Artinian (Noetherian) iff $[X]$ is Artinian (Noetherian).*

Proof. Suppose that $[x_1] \leq [x_2] \leq [x_3] \leq \cdots$. Then $V(x_1) \supseteq V(x_2) \supseteq V(x_3) \supseteq \cdots$. So, there is $k \in \mathbb{N}$ such that $V(x_k) = V(x_n)\ \ \forall n \geq k$. Thus $[x_k] = [x_n]\ \ \forall n \geq k$. Therefore $[X]$ is Artinian. Conversely, let $x_1, x_2, x_3, \cdots$ be elements in X such that $V(x_1) \supseteq V(x_2) \supseteq V(x_3) \supseteq \cdots$. Then $[x_1] \leq [x_2] \leq [x_3] \leq \cdots$. So there is $k \in \mathbb{N}$ such that $[x_n] = [x_k]\ \ \forall n \geq k$. Therefore $V(x_k) = V(x_n)\ \ \forall n \geq k$ and X is Artinian.

Definition 3.3. Suppose that X is an $A-$space. An element $x \in X$ is said to be *maximal* (resp. *minimal*) if $V(x) = V(z)$ whenever $V(z) \subseteq V(x)$ (resp. whenever $V(z) \supseteq V(x)$). We denote the set of maximal elements of X by $M(X)$ (or simply by M), and the set of minimal elements of X by $m(X)$ (or simply by m).

Theorem 3.2. *If X is an Artinian (resp. a Noetherian) $A-$space, then $M \neq \emptyset$ (resp. $m \neq \emptyset$).*

Proof. Suppose that x is not maximal for all $x \in X$. Pick $x_1 \in X$. Then there exists x_2 such that $V(x_2) \subset V(x_2)$ ($:= V(x_2) \subseteq V(x_1)$ and $V(x_2) \neq V(x_1)$). Now x_2 is not maximal, so there exists $x_3 \in X$ such that $V(x_3) \subset V(x_2)$. Continue this process to get $x_1, x_2, x_3, \cdots, x_n, \cdots$ in X such that $V(x_1) \supset V(x_2) \supset V(x_2) \supset \cdots$. Therefore X is not Artinian $A-$space.

Theorem 3.3. *Let X be an Artinian (resp. a Noetherian) $A-$space, then M is open (resp. m is closed) set in X .*

Proof. Let $x \in M$ and let $y \in V(x)$. Then $V(y) \subseteq V(x)$, and so, $V(x) = V(y)$. This implies that $y \in M$. Hence $V(x) \subseteq M$ and M is open.

Corollary 3.4. *If X is an Artinian (resp. a Noetherian) $A-$space, then $x \in M$ iff $[x] \in [M]$. (resp. $x \in m$ iff $[x] \in [m]$.)*

Corollary 3.5. *Let X be an Artinian $A-$space. Then $V(x) \subseteq M$ iff $[V(x)] \subseteq [M]$.*

We will denote the set of maximal (resp. minimal) elements in the shadow space $[X]$ by M_s (resp. m_s)

Theorem 3.6. *If X is an Artinian (resp. a Noetherian) $A-$space, then $[M] = M_s$ (resp. $[m] = m_s$).*

Proof. Let $x \in [M]$. Then there is $a \in M$ such that $x = [a]$. If $[a] \leq [z]$ in $[X]$, then $V(z) \subseteq V(a)$. So $V(z) = V(a)$, and hence $[a] = [z]$. Consequently $x = [a] \in M_s$. On the other hand, let $[x] \in M_s$ and suppose that $V(z) \subseteq V(x)$. Then $[x] \leq [z]$. Since $[x] \in M_s$, $[x] = [z]$. So $V(x) = V(z)$. Hence $x \in M$ and so $[x] \in [M]$.

Definition 3.4. Let X be an $A-$space, $A \subseteq X$, $x \in A$. We say x is *a maximal* in A if x is a maximal in the subspace A. That is, $\forall a \in A,\ V(a) \cap A \subseteq V(x) \cap A$ implies $V(a) \cap A = V(x) \cap A$. We will denote the set of maximal elements of A by $M(A)$ and $V(x) \cap A$ by $V_A(x)$. It is clearly that $V_A(x)$ is the minimal neighbourhood of x in A.

Theorem 3.7. *[5] A subspace of an Artinian T_0 $A-$space is an Artinian T_0 $A-$space.*

Remark 3.1. In general if $A \cap B \subseteq A \cap C$, then B need not be a subset of C. But if X is an $A-$space and $A \subseteq X$ such that $x, y \in A$, then we have that $V_A(x) \subseteq V_A(y)$ iff $V(x) = V(y)$. To see this, if $A \cap V(x) \subseteq A \cap V(y)$, then $x \in V(y)$ and hence $V(x) \subseteq V(y)$.

Theorem 3.8. *A subspcace of an Artinian (resp. a Noetherian) $A-$space is Artinian (resp. Noetherian) $A-$space.*

Proof. Suppose that A is not Artinian. Then there is an infinite sequence of points $x_1, x_2, \cdots, x_n, \cdots$ in A such that $V_A(x_1) \supset V_A(x_2) \supset V_A(x_3) \supset \cdots$. This implies that $V(x_1) \supset V(x_2) \supset V(x_3) \supset \cdots$ in X, and so X is not Artinian.

Remark 3.2. In general, a subspace of a quotient space of a topological space X need not be equal to the quotient space of a subspace of X. But they are equal if the space X is an $A-$space and the quotient space is its corresponding shadow space as shown in the following theorem.

Theorem 3.9. *Let X be an $A-$space and $A \subseteq X$. Then $[A]$ as a subspace of the shadow space $[X]$ is the same as the shadow space of the subspace (A, τ_A).*

Proof. Firstly, $[A]$ is a T_0 Alexandroff space in both cases. Let $([A], \tau(\leq))$ be the subspace of $[X]$ and $([A], \tau_s(\leq_s))$ is the shadow space of the subspace (A, τ_A). For $a, b \in A$, $[a] \leq [b]$ in $([A], \tau(\leq))$ iff $[a] \leq [b]$ in $[X]$ iff $V(a) \supseteq V(b)$ in X iff $V_A(a) \supseteq V_A(b)$ in (A, τ_A) iff $[a] \leq_s [b]$ in $([A], \tau_s(\leq_s))$.

Theorem 3.10. *Let X be an Artinian $A-$space and $A \subseteq X$. Then $[M(A)] = M[A]$.*

Proof. If $[x] \in [M(A)]$, then there exists $y \in M(A)$ such that $[x] = [y]$. Let $[z] \in [A]$ such that $[z] \geq [y]$. Then $V_A(y) \supseteq V_A(z)$. So $V_A(z) = V_A(y)$, and so $[y] = [z]$. Hence $[x] \in M[A]$. Convesely, let $[x] \in M[A]$ and let $z \in A$ such that $V_A(z) \subseteq V_A(x)$. Then $[x] \leq [z]$ and so $[x] = [z]$ in $[A]$. Then $V_A(z) = V_A(x)$. Hence $x \in M(A)$ and $[x] \in [M(A)]$.

Theorem 3.11. *Let X be an Artinian $A-$space and let $A \subseteq X$. Then $Cl(A) = Cl(M(A))$.*

Proof. If $c \in Cl(A)$, then $[c] \in [Cl(A)] = Cl([A])$. So $[c] \in \downarrow M[A]$ and there exists $[a] \in M[A]$ such that $[c] \leq [a]$. This implies that there exists $b \in M(A)$ such that $V(b) \subseteq V(c)$. Therefore $c \in Cl(M(A))$. The converse is obvious.

Corollary 3.12. *Let X be an Artinian $A-$space, and let $A \subseteq X$. If A is open, then $M(A) \subseteq M$.*

Proof. Let $y \in M(A)$, and let $x \in X$ be such that $V(x) \subseteq V(y)$. Then $[y] \in M[A]$ and $[y] \leq [x]$. Since $[A]$ is open , $[x] \in [A]$, and $[y] = [x]$. So $V(x) = V(y)$, and hence $y \in M$.

Definition 3.5. Let X be an $A-$space, we define $\widehat{x} = V(x) \cap M$ and $\check{x} = \overline{x} \cap m$.

Theorem 3.13. *If X is an Artinian (resp. a Noetherian) $A-$space, then $\widehat{x} \neq \emptyset$ (resp. $\check{x} \neq \emptyset$) $\forall x \in X$.*

Proof. Suppose that, there is $x \in X$ such that $V(x) \cap M = \emptyset$. Then $V(x) \subseteq M^c$ and x is not maximal. So there is $x_1 \in X$ such that $V(x_1) \subset V(x)$. Again $x_1 \in M^c$, so there is $x_2 \in X$ such that $V(x_2) \subset V(x_1)$. Continue this process to get, $x, x_1, x_2, x_3, \cdots$ such that $V(x) \supset V(x_1) \supset V(x_2) \supset V(x_3) \supset \cdots$. Therefore X is not Artinian .

Theorem 3.14. *If X is an Artinian (resp. a Noetherian) Alexandroff space then $[\widehat{x}] = \widehat{[x]}$ (resp. $[\check{x}] = \check{[x]}$).*

Proof. $[\widehat{x}] = [V(x) \cap M] = [V(x)] \cap [M] = \uparrow [x] \cap M_s = \widehat{[x]}$.

Theorem 3.15. *Let X be an Artinian $A-$space, and let $A \subseteq X$. If A is open (closed) then $\widehat{x} \subseteq A$ (resp. $\check{x} \subseteq A$) for all $x \in A$.*

Proof. Let $x \in A$ and let $y \in \widehat{x}$. A is open implies that $[A]$ is open in $([X], \tau_s)$ and $\widehat{[x]} \subseteq [A]$. Then $[y] \in [\widehat{x}] = \widehat{[x]} \subseteq [A]$. By Theorem 2.4, $y \in A$.

Corollary 3.16. *Let X be an Artinian $A-$space and $A \subseteq X$. If $A \cap M = \emptyset$, then $Int(A) = \emptyset$.*

Proof. If $x \in Int(A)$, then by Theorem 3.15. $\widehat{x} \subseteq Int(A)$. Since $\widehat{x} \neq \emptyset$, $\widehat{x} \subseteq M$, we get $\widehat{x} \subseteq Int(A) \cap M$. Thus $A \cap M \neq \emptyset$.

Theorem 3.17. *Let X be an Artinian $A-$space. If $M \subseteq A$, then A is dense in X.*

Proof. Since $M \subseteq A$, $[M] \subseteq [A]$. So by Theorem 2.1(4), $[A]$ is dense in $[X]$. Thus by Theorem 2.3, A is dense in X .

The converse is not always true as shown in the following example.

Example 3.18. *Let $X = \{1, 2, 3, 4, 5\}$ and let $\tau = \{X, \emptyset, \{3\}, \{4, 5\}, \{3, 4, 5\}\}$, then $M = \{3, 4, 5\}$. Let $A = \{1, 2, 3, 5\}$ then $cl(A) = X$. But $4 \in M$ and $4 \notin A$.*

Theorem 3.19. *Let X be an Artinian $A-$space, and $A \subseteq X$. Then A is nowhere dense iff $A \cap M = \emptyset$.*

Proof. If $x \in A \cap M$, then $x \in Cl(A)$ and $x \in M$. Let $y \in V(x)$, then $V(y) = V(x)$. So $V(y) \cap A \neq \emptyset$ and hence $y \in Cl(A)$. That is $V(x) \subseteq Cl(A)$, and therefore $x \in Int(Cl(A))$. Conversely, if $M \cap A = \emptyset$, then $[M \cap A] = \emptyset$. So $[M] \cap [A] = \emptyset$. By Theorem 2.1 part (5), $Int_s(Cl_s([A])) = \emptyset$. Thus $[Int(Cl(A))] \subseteq Int_s([Cl(A)]) = \emptyset$. Therefore $Int(Cl(A)) = \emptyset$.

Theorem 3.20. *Let X be an $A-$space, and let $A \subseteq X$. Then A is nowhere dense iff $[A]$ is nowhere dense.*

Proof. Let $[x] \in Int_s(Cl_s([A]))$. Then $[x] \in Int_s([Cl(A)])$, and so $\uparrow [x] \subseteq [Cl(A)]$. Equivalently $[x] \in [V(x)] \subseteq [Cl(A)]$. Hence $x \in V(x) \subseteq Cl(A)$, and $x \in Int(Cl(A))$. Conversely, if $Int_s(Cl_s([A])) = \emptyset$, then $Int_s([Cl(A)]) = \emptyset$. Hence $[Int(Cl(A))] \subseteq Int_s([Cl(A)]) = \emptyset$, and $Int(Cl(A)) = \emptyset$.

Theorem 3.21. *Let X be an $A-$space. If $V(x) = V(y)\ \forall x, y \in M$, then every subset of X is either dense or nowhere dense.*

Proof. Suppose that $V(x) = V(y)$ for all $x, y \in M$. Then $[x] = [y]$ for all $x, y \in M$. Hence $|[M]| = 1$. By Theorem 2.1 part (6) every subset of $[X]$ is either dense or nowhere dense. Let $A \subseteq X$. Then $[A]$ is either dense or nowhere dense. By Theorem 2.3 and Theorem 3.20, A is dense or nowhere dense.

4 Connectedness, Compactness and Resolvability Properties

Theorem 4.1. *An $A-$space X is compact iff $[X]$ is compact.*

Proof. A quotient space of compact space is compact. Conversely, suppose that $[X]$ is compact, and let $\{u_\alpha : \alpha \in \Delta\}$ be an open cover of X. Then $\{[u_\alpha] : \alpha \in \Delta\}$ is an open cover of $[X]$, so there is a finite subcover of $[X]$, $\{[u_{\alpha_i}] : 1 \leq i \leq n\}$. By Theorem 2.4, $\{u_{\alpha\, i}, 1 \leq i \leq n\}$ is a finite subcover of X.

Theorem 4.2. *An $A-$space X is connected iff $[X]$ is connected.*

Proof. A quotient space of a connected space is connected. Conversely if there is nonempty disjoint open sets A, B such that $X = A \cup B$. Therefore $[A] \cap [B] = \emptyset$ and $[X] = [A \cup B] = [A] \cup [B]$.

Theorem 4.3. *Let X be an $A-$space, and $A \subseteq X$. If x is an isolated point of A, then $[x]$ is an isolated point of $[A]$.*

Example 4.4. *Recall Example 3.18. Let $A = \{1, 4, 5\}$, then A has no isolated points. But $\{[4]\}$ is an isolated point of $[A]$.*

Theorem 4.5. *Let X be an $A-$space, and let $A \subseteq X$. Then $[A]' \subseteq [A']$.*

Proof. If $[x] \in Cl_s([A])$ and $[x]$ is not an isolated point of $[A]$, then $[x] \in [Cl(A)]$ and $[x]$ is not an isolated point of $[A]$. So $x \in Cl(A)$ and x is not isolated point of A.

Definition 4.1. A topological space X is *scattered* if no subset of X is dense in itself and α- *scattered* if it has a dense subset of isolated points.

Theorem 4.6. *[2] An Artinian T_0 $A-$space X is scattered and α- scattered.*

In general an $A-$space need not be scattered and need not be α- scattered, as shown in the following example.

Example 4.7. *The set A in Example 4.4 is dense in itself . So X is not scattered. Moreover the only isolated point in X is $x = 3$ and $\{3\}$ is not dense in X. So X is also not α- scattered.*

Definition 4.2. [6] A topological space X is called *resolvable* iff $X = D \cup D^c$ where both D and D^c are dense. If X is not resolvable, it is *irresolvable*. X is called *hereditarily irresolvable*, if no non empty subset is resolvable .

An Artinian T_0 $A-$space is always irresolvable [2]. In fact it is hereditarily irresolvable. In the following two theorems, we give characterizations for an Artinian $A-$space to be irresolvable and hereditarily irresolvable.

Theorem 4.8. *An Artinian $A-$space X is irresolvable iff X contains isolated points.*

Proof. Suppose that $\{x\}$ is not open for all $x \in X$. Then $\forall x \in X$, $|V(x)| \geq 2$. So $\forall x \in M$, pick a_x, b_x from $V(x)$ such that $a_x \neq b_x$. Set $D_1 = \{a_x : x \in M\}$, $D_2 = \{b_x : x \in M\}$. Then we have that $[D_1] = [D_2] = [M]$ which is dense in $[X]$. By Theorem 2.3, both D_1 and D_2 are dense in X. Moreover $D_1 \cap D_2 = \emptyset$, so $D_2 \subseteq D_1^c$ and hence D_1^c is also dence in X. Therefore X is irresolvable. Conversely, if there exist $x \in X$ such that $\{x\}$ is open, then $\{x\}$ is a subset of every dense subset of X. So there is no disjoint dense subsets of X. ∎

Theorem 4.9. *An Artinian $A-$space X is hereditarily irresolvable iff X is T_0.*

Proof. If there exist $x \neq y$ in X such that x, y can't be separated, then $V(x) = V(y)$. Let $A = \{x, y\}$. Then $\{x\}, \{y\}$ are two dense sets in A such that $A = \{x\} \cup \{y\}$. Then A is resolvable. For the converse see [2].

5 Nearly Open Sets

Definition 5.1. A subset A of a topological space (X, τ) is called

(1) *a semi-open set* [7] if $A \subseteq Cl(Int(A))$.

(2) *a preopen* set [8] if $A \subseteq Int(Cl(A))$.

(3) an α *open* set [9] if $A \subseteq Int(Cl(Int(A)))$.

The family of all semi-open (resp. preopen, α open) sets is denoted by SO(X) (resp. PO(X), τ_α).

Theorem 5.1. *Let X be an $A-$space, and let $A \subseteq X$. If A is semi-open in X, then $[A]$ is semi-open in $[X]$.*

Proof. Let $A \subseteq Cl(Int(A))$. Then $[A] \subseteq [Cl(Int(A))] = Cl_s([Int(A)]) \subseteq Cl_s(Int_s([A]))$.

The converse is not always true, as shown in the following example.

Example 5.2. *In Example 3.18, $\tau_s = \{[X], \emptyset, \{[3]\}, \{[4]\}, \{[3], [4]\}\}$. Let $A = \{1, 2, 5\}$, then $Int(A) = \emptyset$ and so $Cl(Int(A)) = \emptyset$. But $[A] = \{[1], [4]\}$ and $Cl_s(Int_s([A])) = \{[1], [4]\}$. Thus $[A]$ is semi-open in $[X]$, but A is not semi-open in X.*

Theorem 5.3. *Let X be an $A-$space, and let $A \subseteq X$. Then A is preopen iff $[A]$ is preopen.*

Proof. If $A \subseteq X$ such that $A \subseteq Int(Cl(A))$, then $[A] \subseteq [Int(Cl(A))] \subseteq Int_s[(Cl(A))] = Int_s(Cl_s([A]))$. Conversely, let $x \in A$. So $[x] \in [A] \subseteq Int_s(Cl_s([A]))$. Consequently $[x] \in \uparrow [x] \subseteq Cl_s([A]) = [Cl(A)]$. By Theorem 2.4, $x \in V(x) \subseteq Cl(A)$. Thus $x \in Int(Cl(A))$.

Theorem 5.4. *Let X be an $A-$space, and $A \subseteq X$. If A is $\alpha - open$, then $[A]$ is $\alpha - open$.*

Proof. If A is $\alpha - open$, then A is both semi-open and preopen. Thus $[A]$ is both semi-open and preopen, and hence $\alpha - open$.

The converse is not always true as shown in the following example.

Example 5.5. *Recall Example 3.18 Let $A = \{4\}$. Then $[A] = \{[4]\} = \{\{4, 5\}\}$. Then $Int(Cl(Int(A))) = \emptyset$ and $Int_s(Cl_s(Int_s([A]))) = \{[4]\}$. So $[A] \subseteq Int_s(Cl_(Int_s([A])))$, but A is not a subset of $Int(Cl(Int(A)))$.*

Theorem 5.6. *[2] Let X be an Artinian T_0 $A-$space, and let $A \subseteq X$. Then A is semi-open, iff $M(A) \subseteq M$.*

Theorem 5.7. *Let X be an Artinian $A-$space, and let $A \subseteq X$. If A is semi-open, then $M(A) \subseteq M$.*

Proof. If A is semi-open, then $[A]$ is semi-open. By Theorem 5.6, $M[A] \subseteq [M]$, so $[M(A)] \subseteq [M]$. If $x \in M(A)$, then $[x] \in [M(A)]$ which implies that $[x] \in [M]$. By Theorem 2.4, $x \in M$.

Theorem 5.8. *[2] Let X be an Artinian T_0 $A-$space, and let $A \subseteq X$. If A is preopen, then $M(A) \subseteq M$.*

Theorem 5.9. *Let X be an Artinian $A-$space, and let $A \subseteq X$. If A is preopen, then $M(A) \subseteq M$.*

Proof. If A is preopen, then $[A]$ is preopen. So by Theorem 5.8 $M[A] \subseteq [M]$, and so $[M(A)] \subseteq [M]$. If $x \in M(A)$, then $[x] \in [M(A)]$. Hence $[x] \in [M]$, and $x \in M$.

The converse of Theorem 5.7 and 5.9 are not always true, as shown in the following example.

Example 5.10. *In Example 3.18, if $A = \{1, 2, 5\}$, then $M(A) = \{5\}$ and $M = \{3, 4, 5\}$. So $M(A) \subseteq M$. But A is neither semi-open nor preopen.*

Theorem 5.11. *[2] Let X be an Artinian T_0 Alexandroff space, then*

(1) $PO(X) \subseteq SO(X)$.

(2) $PO(X) = \tau_\alpha$.

Example 5.12. *In Example 3.18, let $A = \{4\}$, then A is preopen but not $\alpha - open$ and not semi-open. So $PO(X)$ is not a subclass of $SO(X)$ and $PO(X) \neq \tau_\alpha$. So, the results in Theorem 5.11 for Artinian T_0 $A-$space need not be true in general for non T_0 Artinian $A-$spaces.*

6 Conclusions

Relation between $A-$spaces and their shadow spaces is very interesting in Alexandroff literature. In this paper, we use this relation in introducing new definitions and concepts such as Artinian, Noetherian, ACC, etc., defined on $A-$shadow spaces and carry over to any $A-$spaces. These concepts on shadow spaces can be translated into corresponding posets. This process proves to be an easier approach. In this regard, we think that this paper is useful and important in the direction of clarification of $A-$spaces.

Competing Interests

The authors declare that no competing interests exist.

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