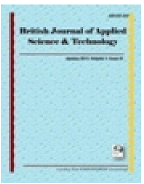

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On Some Generalizations via Multinomial Coefficients

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Authors' contributions

This work was carried out in collaboration between all authors. Author MM designed the idea of the study and managed the literature searches. Author NP obtained the results and wrote the first draft of the manuscript. Author MM proof read and modified the manuscript. Authors MM and NP carried out the revision of the manuscript and the additional literature searches. All authors read and approved the final manuscript.

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Abstract

This paper gives a brief discussion on the Multinomial coefficients. Using this notion, we obtain generalizations of the Vandermonde's and the Chu Shih-Chieh's identities for the Binomial coefficients, respectively. This is done through the use of two known principles in Combinatorics, namely, the Addition and the Multiplication principles. Some examples of generating functions of a sequence involving the multinomial coefficients are also derived and presented.

Keywords: Binomial coefficients; Multinomial coefficients; Generating functions; q-analogues

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1 Introduction

For a set A of n distinct objects, a *combination* of A is simply a subset of A. More precisely, for $0 \leq r \leq n$, an r-*combination* of A is an r-element subset of A (see [1]). From here, the *binomial coefficients* $\binom{n}{r}$, read as "n taken r", is defined to be the number of r-combinations of the set A. Appearently, the numbers $\binom{n}{r}$ play an important role in enumerative combinatorics and other field of discipline. One may see the books by Chen and Kho [1] and Comtet [2] for a more detailed discussion on the binomial coefficients. We note that $\binom{n}{r}$ can be expressed explicitly as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{(n)_r}{r!}, \tag{1.1}$$

where $(n)_r = n(n-1)(n-2)\ldots(n-r+1)$ is the falling factorial of n of order r and $r! = r(r-1)(r-2)\ldots(3)(2)(1)$. The term "binomial coefficients" comes from the fact that the numbers $\binom{n}{r}$ appear as coefficients in the expansion of the binomial expression $(x+y)^n$ as seen in the well-known *Binomial theorem*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{1.2}$$

When $y = 1$, (1.2) becomes

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r, \tag{1.3}$$

which is the ordinary generating function of the binomial coefficients. Other basic properties and identities involving the binomial coefficients are the following:

- the triangular recurrence relation

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}; \tag{1.4}$$

- the identities

$$\binom{n}{r} = \binom{n}{n-r}; \tag{1.5}$$

  and

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n. \tag{1.6}$$

Combinatorially, the binomial coefficients $\binom{n}{r}$ is interpreted as the number of ways to distribute r identical objects into n distinct boxes such that each box can hold at most one object. In 1772, A. T. Vandermonde obtained the next identity which is now popularly known as the *Vandermonde's identity* given by

$$\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}, \tag{1.7}$$

where n, m, and r are positive integers. Other known results are the *Chu Shih-Chieh's identities*

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{n}{r}, \tag{1.8}$$

for all positive integers r and n with $n \geq r$; and

$$\binom{r+k+1}{k} = \binom{r}{0} + \binom{r+1}{1} + \ldots + \binom{r+k}{k}, \tag{1.9}$$

for all positive integers r and k, discovered by Chu Shih-Chieh in 1303. Equations (1.9) and (1.8) are often referred to as the *Hockey-Stick identities*. The equations (1.7), (1.9) and (1.8) can be found in [1].

On the otherhand, the *Multinomial coefficients*

$$\binom{n}{n_1, n_2, \ldots, n_m} \tag{1.10}$$

is an identity which generalizes the binomial coefficients $\binom{n}{r}$. The multinomial coefficients count the number of ways to distribute n distinct objects into m distinct boxes such that n_1 of them are in box 1, n_2 of them are in box 2, ..., and n_m of them are in box m, where n, m, n_i, $i = 1, 2, \ldots, m$, are non-negative integers such that $n_1 + n_2 + \ldots + n_m = n$. Using this interpretation, it is easy to show that the multinomial coefficients satisfy the explicit form

$$\binom{n}{n_1, n_2, \ldots, n_m} = \frac{n!}{n_1!n_2!\ldots n_m!}. \tag{1.11}$$

Clearly, when $m = 2$ in (1.11),

$$\binom{n}{n_1, n_2} = \frac{n!}{n_1!n_2!} = \binom{n}{n_1},\ n_2 = n - n_1,$$

which is the binomial coefficients. The multinomial coefficients $\binom{n}{n_1, n_2, \ldots, n_m}$ are known to have the following combinatorial interpretations:

- the number of ways to partition an n-element set X into m parts P_1, P_2, ..., P_m such that $|P_1| = n_1$, $|P_2| = n_2$, ..., $|P_m| = n_m$ with $n_1 + n_2 + \ldots + n_m$; and
- the number of permutations of n objects (not necessarily distinct) taken all at a time. This is equivalent to the number of ways to arrange n objects in a row.

The study of the binomial and the multinomial coefficients as well as their different extensions and applications is popular among mathematicians (*e.g.* [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and some of the references therein). Among the known properties of the multinomial coefficients are the following:

- the Multinomial theorem

$$(x_1 + x_2 + \ldots + x_m)^n = \sum_{0 \leq n_1, n_2, \ldots, n_m \leq n} \binom{n}{n_1, n_2, \ldots, n_m} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}, \tag{1.12}$$

  for positive integers n and m and $\sum_{i=1}^{m} n_i = n$;
- the identities

$$\binom{n}{n_1, n_2, \ldots, n_m} = \binom{n}{n_{\alpha(1)}, n_{\alpha(2)}, \ldots, n_{\alpha(m)}}, \tag{1.13}$$

  where $\{\alpha(1), \alpha(2), \ldots, \alpha(m)\} = \{1, 2, \ldots, m\}$,

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_m} &= \binom{n-1}{n_1 - 1, n_2, \ldots, n_m} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_m} \\ &\quad + \ldots + \binom{n-1}{n_1, n_2, \ldots, n_m - 1};\end{aligned} \tag{1.14}$$

  and

$$\sum_{0 \leq n_1, n_2, \ldots, n_m \leq n} \binom{n}{n_1, n_2, \ldots, n_m} = m^n. \tag{1.15}$$

Note that the binomial identities (1.2), (1.5), (1.4), and (1.6) can be obtained from the multinomial identities (1.12), (1.13), (1.14), and (1.15), respectively when $m = 2$. This motivates us to establish more properties and identities involving the multinomial coefficients that will generalize the results in the binomial coefficients. In order to achieve some of the main results of this paper, we will make use of the Addition and the Multiplication principles stated as follow:

- ***Addition Principle*** [1]. Assume that there are n_1 ways for the event E_1 to occur, n_2 ways for the event E_2 to occur, ..., n_k ways for the event E_k to occur, where $k \geq 1$. If these ways for the different events to occur are pairwise disjoint, then the number of ways for atleast one of these events E_1, E_2, ..., or E_k to occur is
$$n_1 + n_2 + \ldots + n_k = \sum_{i=1}^{k} n_i.$$
- ***Multiplication Principle*** [1]. Assume that an event E can be decomposed into r ordered events $E_1, E_2, \ldots, E_r$ and that there are E_1 to occur, n_2 ways for the event E_2 to occur, ..., n_r ways for the event E_r to occur. Then the total number of ways for the event E to occur is given by
$$n_1 \times n_2 \times \ldots \times n_r = \prod_{i=1}^{r} n_i.$$

The Addition and the Multiplication principles are two of the many fundamental tools used in proving combinatorial identities. For a more detailed discussion on these principles, see [1].

The results of this paper are organized as follow: in section 2, a formula that will generalize the Vandermonde's identity in (1.7) is derived in terms of the multinomial coefficients; in section 3, some identities that will generalize the Chu Shih-Chieh's identities in (1.9) and (1.8) are presented; and in section 4, the generating functions of a sequence involving the multinomial coefficients is examined.

2 Generalized Vandermonde's Identity

In this section, we will derive a generalization of the Vandermonde's identity in (1.7) in terms of the multinomial coefficients. To achieve this, we first let E be the event of distributing n distinct objects to k distinct boxes such that box 1 contains n_1 objects, box 2 contains n_2 objects, ..., and box k contains n_k objects so that
$$|E| = \binom{n}{n_1, n_2, \ldots, n_k}.$$
For a non-negative integer r, where $r \leq n$, event E occurs if the two succeeding events E_1 and E_2 occur:

E_1 := the event of distributing the first r objects to the k boxes so that box 1 contains at most n_1 objects, box 2 contains at most n_2 objects, ..., and box k contains at most n_k objects. That is, box 1 contains r_1 objects, box 2 contains r_2 objects, ..., and box k contains r_k objects, where $r_i \leq n_i$ for $i = 1, 2, \ldots k$; and

E_2 := the event of distributing the remaining $n - r$ objects to the k boxes so that $n_1 - r_1$ objects will be placed in box 1, $n_2 - r_2$ objects will be placed in box 2, ..., and $n_k - r_k$ objects will be placed in box k.

Now, given a k-ary sequence $(r_1, r_2, \ldots, r_k)$ of non-negative integers with $\sum_{j=1}^{k} r_j = r$, the event E_1 occurs in $\binom{r}{r_1, r_2, \ldots, r_k}$ ways while the event E_2 occurs in $\binom{n-r}{n_1-r_1, n_2-r_2, \ldots, n_k-r_k}$ ways. Moreover, by Multiplication principle, the two succeeding events occur in
$$\binom{r}{r_1, r_2, \ldots, r_k}\binom{n-r}{n_1 - r_1, n_2 - r_2, \ldots, n_k - r_k}$$

ways. Note that event E occurs for any k-ary sequence $(r_1, r_2, \dots, r_k)$ of non-negative integers with $\sum_{j=1}^{k} r_j = r$. Hence, by Addition principle,

$$|E| = \sum_{r_1+r_2+\dots+r_k=r} \binom{r}{r_1, r_2, \dots, r_k} \binom{n-r}{n_1-r_1, n_2-r_2, \dots, n_k-r_k}. \tag{2.1}$$

We will state this result in the following theorem.

Theorem 2.1. *Let n and r be positive integers such that $\sum_{j=1}^{k} n_j = n$. Then*

$$\binom{n}{n_1, n_2, \dots, n_k} = \sum_{r_1+r_2+\dots+r_k=r} \binom{r}{r_1, r_2, \dots, r_k} \binom{n-r}{n_1-r_1, n_2-r_2, \dots, n_k-r_k}, \tag{2.2}$$

where the sum is taken over all k-ary sequences $(r_1, r_2, \dots, r_k)$ of non-negative integers, where $\sum_{j=1}^{k} r_j = r$.

Remark 2.1. When $k = 2$ in (2.2), we have

$$\binom{n}{n_1} = \sum_{r_1=0}^{n_1} \binom{r}{r_1} \binom{n-r}{n_1-r_1}.$$

This is exactly the Vandermonde's identity in (1.7). Also, when $n = r + m$ (m is a positive integer), then (2.2) becomes

$$\binom{r+m}{n_1, n_2, \dots, n_k} = \sum_{r_1+r_2+\dots+r_k=r} \binom{r}{r_1, r_2, \dots, r_k} \binom{m}{n_1-r_1, n_2-r_2, \dots, n_k-r_k}, \tag{2.3}$$

where $\sum_{j=1}^{k} n_j = r + m$. (2.3) is actually identical to the formula which was earlier considered by Tauber [3] and Carlitz [4].

To illustrate (2.3), we consider the following basic problem in distribution.

Example 2.2. *Suppose that a college professor wanted to form 3 teams from a group of 5 top female and 2 top male students coming from different basic Math classes. In how many ways can this be done if the said professor added a condition that the first team should have 2 members, the second team should have 4 members and the third team should have 1 member only?*

To solve this, note that from (2.3)*, we have*

$$\binom{5+2}{2,4,1} = \sum_{r_1+r_2+r_3=5} \binom{5}{r_1, r_2, r_3} \binom{2}{2-r_1, 4-r_2, 1-r_3}. \tag{2.4}$$

Observe that in order for the coefficient $\binom{2}{2-r_1,4-r_2,1-r_3}$ to exist, we must have $r_1 \le 2, r_2 \le 4, r_3 \le 1$. Hence, the possible values of r_1, r_2, r_3 for which $\binom{5}{r_1,r_2,r_3}$ is conformable with $\binom{2}{2-r_1,4-r_2,1-r_3}$ are the following:

$$\binom{5}{2,2,1}, \binom{5}{2,3,0}, \binom{5}{1,4,0}, \binom{5}{1,3,1}, \binom{5}{0,4,1}.$$

Furthermore, we have

$$\begin{aligned}
\binom{5+2}{2,4,1} &= \binom{5}{2,2,1}\binom{2}{0,2,0} + \binom{5}{2,3,0}\binom{2}{0,1,1} + \binom{5}{1,4,0}\binom{2}{1,0,1} \\
&\quad + \binom{5}{1,3,1}\binom{2}{1,1,0} + \binom{5}{0,4,1}\binom{2}{2,0,0} \\
&= 30 + 20 + 10 + 40 + 5 \\
&= 105.
\end{aligned}$$

Thus, there are 105 ways.

3 Generalized Chu Shih-Chieh's Identities

The theorems in this section contain a generalized version of the Chu Shih-Chieh's identities in (1.9) and (1.8) in terms of the multinomial coefficients.

Theorem 3.1. *Let r and m be non-negative integers. For any k-ary sequence $(r_1, r_2, \dots, r_k)$ of non-negative integers with $\sum_{j=1}^{k} r_j = r$,*

$$\binom{r+m+1}{r_1, r_2, \dots, r_k, m+1} = \sum_{i=0}^{r} \sum \binom{i}{i_1, i_2, \dots, i_k, 0} \binom{r+m-i}{r_1-i_1, r_2-i_2, \dots, r_k-i_k, m}, \tag{3.1}$$

where the inner sum is taken over all k-ary sequences $(i_1, i_2, \dots, i_k)$ of non-negative integers with $\sum_{j=1}^{k} i_j = i$ and $i_j \le r_j$ for $j = 1, 2, \dots k$.

Remark 3.1. When $k = 1$, (3.1) becomes

$$\binom{r+m+1}{r} = \sum_{i=0}^{r} \binom{r+m-i}{r-i}.$$

This is identical to the first Chu Shih-Chieh's identity in (1.9).

Proof of Theorem 3.1. Let $S = \{1, 2, \dots, r+m+1\}$ with $|S| = r+m+1$ and let E be the event of distributing the elements of S to $k+1$ disjoint subsets $S_1, S_2, \dots, S_{k+1}$ of S so that $|S_1| = r_1, |S_2| = r_2, \dots, |S_k| = r_k$ and $|S_{k+1}| = m+1$. Hence

$$|E| = \binom{r+m+1}{r_1, r_2, \dots, r_k, m+1}.$$

We may also count $|E|$ as follows.

Note that event E occurs if any of the $r+1$ disjoint events $E_0, E_1, E_2, \dots, E_r$ occur, where

E_i := the event of distributing the elements of S to $k+1$ disjoint subsets as stated in E, where $1, 2, \dots, i \notin S_{k+1}$ and $i+1 \in S_{k+1}$ for $i = 0, 1, 2, \dots, r$.

Now, consider a k-ary sequence $(i_1, i_2, \dots, i_k)$ of non-negative integers with $\sum_{j=1}^{k} i_j = i$ and $i_j \le r_j$ for $j = 1, 2, \dots k$, we may decompose each of the events E_i's into the following events:

E_{i_1} := the event of distributing the elements $1, 2, \dots, i$ to the $k+1$ subsets so that i_1 of them is placed in S_1, i_2 of them are placed in S_2, ..., i_k of them are placed in S_k, and none is placed in S_{k+1} so that

$$|E_{i_1}| = \binom{i}{i_1, i_2, \dots, i_k, 0}.$$

E_{i_2} := the event of placing the element $i+1$ in S_{k+1} so that $|E_{i_2}| = 1$.

E_{i_3} := the event of filling up the subsets with the remaining elements $i+2, i+3, \dots, r+m+1$ so that $|S_1| = r_1, |S_2| = r_2, \dots, |S_k| = r_k$ and $|S_{k+1}| = m+1$. That is, from the remaining $r+m-i$ elements, we place $r_1 - i_1$ elements in S_1, $r_2 - i_2$ elements in S_2, ..., $r_k - i_k$ elements in S_k, and m elements in S_{k+1}. Thus,

$$|E_{i_3}| = \binom{r+m-i}{r_1-i_1, r_2-i_2, \dots, r_k-i_k, m}.$$

Hence, by Multiplication principle, the number of ways of distributing the elements of S to the $k+1$ subsets so that $|S_1| = r_1, |S_2| = r_2, \ldots, |S_k| = r_k$ and $|S_{k+1}| = m+1$, where $1, 2, \ldots, i \notin S_{k+1}$ and $i+1 \in S_{k+1}$ for the k-ary sequence $(i_1, i_2, \ldots, i_k)$ of non-negative integers with $\sum_{j=1}^{k} i_j = i$ and $i_j \leq r_j$ for $j = 1, 2, \ldots k$ is

$$\prod_{j=1}^{3} |E_{i_j}| = \binom{i}{i_1, i_2, \ldots, i_k, 0}\binom{r+m-i}{r_1 - i_1, r_2 - i_2, \ldots, r_k - i_k, m}.$$

Now, considering all k-ary sequences $(i_1, i_2, \ldots, i_k)$ of non-negative integers with $\sum_{j=1}^{k} i_j = i$ and $i_j \leq r_j$ for $j = 1, 2, \ldots k$, by Addition principle,

$$|E_i| = \sum_{i_1+i_2+\ldots+i_k=i} \binom{i}{i_1, i_2, \ldots, i_k, 0}\binom{r+m-i}{r_1 - i_1, r_2 - i_2, \ldots, r_k - i_k, m}.$$

Applying the Addition principle,

$$|E| = \sum_{i=0}^{r} |E_i|.$$

Hence, the proof is done. □

If $S = \{1, 2, \ldots, r+m+1\}$ is a set with $|S| = r+m+1$ and E is the event of distributing the elements of S to $k+1$ disjoint subsets $S_1, S_2, \ldots, S_{k+1}$ of S so that $|S_1| = r_1, |S_2| = r_2, \ldots, |S_k| = r_k$ and $|S_{k+1}| = m+1$, that is,

$$|E| = \binom{r+m+1}{r_1, r_2, \ldots, r_k, m+1},$$

then may also count $|E|$ in the following manner:

Note that E occurs if any of the $m+2$ disjoint events $E_0, E_1, E_2, \ldots, E_{m+1}$ occur,

E_i := the event of distributing the elements of S to $k+1$ disjoint subsets as stated in E, where $1, 2, \ldots, i \in S_{k+1}$ and $i+1 \notin S_{k+1}$ for $i = 0, 1, 2, \ldots, m+1$.

It can be observed that each event E_i occurs if any of the k disjoint events $E_{i1}, E_{i2}, \ldots, E_{ik}$ occur where

E_{ij} := the event of distributing the elements of S to the $k+1$ disjoint subsets as stated in E, where $1, 2, \ldots, i \in S_{k+1}$ and $i+1 \in S_j$ for $j = 1, 2, \ldots, k$ so that

$$|E_{ij}| = \binom{r+m-i}{r_1, r_2, \ldots, r_{j-1}, \ldots, r_k, m-i+1}.$$

Hence, by Addition principle,

$$\begin{aligned} |E_i| &= \sum_{j=1}^{k} |E_{ij}| \\ &= \binom{r+m-i}{r_1 - 1, r_2, \ldots, r_k, m-i+1} + \binom{r+m-i}{r_1, r_2 - 1, \ldots, r_k, m-i+1} \\ &\quad + \ldots + \binom{r+m-i}{r_1, r_2, \ldots, r_k - 1, m-i+1}. \end{aligned}$$

Moreover, we have

$$|E| = \sum_{i=0}^{r} |E_i|.$$

This result is embedded in the next theorem.

Theorem 3.2. *For non-negative integers r and m, and for a k-ary sequence $(r_1, r_2, \ldots, r_k)$ of non-negative integers with $\sum_{j=1}^{k} r_j = r$,*

$$\binom{r+m+1}{r_1, r_2, \ldots, r_k, m+1} = \sum_{i=0}^{m+1} \left[\binom{r+i-1}{r_1-1, r_2, \ldots, r_k, i} + \binom{r+i-1}{r_1, r_2-1, \ldots, r_k, i} + \ldots + \binom{r+i-1}{r_1, r_2, \ldots, r_k-1, i} \right]. \quad (3.2)$$

Remark 3.2. Clearly, it can be verified that when $k = 1$ in (3.2), we recover the second Chu Shih-Chieh's identity in (1.8).

Note that the famous *Pascal's Triangle* in Figure 1 can be expressed via multinomial coefficients as seen in Figure 2.

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array}$$

Figure 1: Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0,0} & & & & & \\ & & & & \binom{1}{0,1} & & \binom{1}{1,0} & & & & \\ & & & \binom{2}{0,2} & & \binom{2}{1,1} & & \binom{2}{2,0} & & & \\ & & \binom{3}{0,3} & & \binom{3}{1,2} & & \binom{3}{2,1} & & \binom{3}{3,0} & & \\ & \binom{4}{0,4} & & \binom{4}{1,3} & & \binom{4}{2,2} & & \binom{4}{3,1} & & \binom{4}{4,0} & \\ \binom{5}{0,5} & & \binom{5}{1,4} & & \binom{5}{2,3} & & \binom{5}{3,2} & & \binom{5}{4,1} & & \binom{5}{5,0} \end{array}$$

Figure 2: Pascal's Triangle (in multinomial coefficients)

Notice that Figure 3 gives a simple illustration of (3.2) since

$$\binom{5}{2,3} = \binom{1}{1,0} + \binom{2}{1,1} + \binom{3}{1,2} + \binom{4}{1,3}$$

is the case when $r = 2$, $m = 2$ and $k = 1$. Similarly, it can be seen in Figure 4 that

$$\binom{5}{4,1} = \binom{4}{4,0} + \binom{3}{3,0} + \binom{2}{2,0} + \binom{1}{1,0} + \binom{0}{0,0},$$

which is precisely (3.1), where $r = 4$, $m = 0$ and $k = 1$.

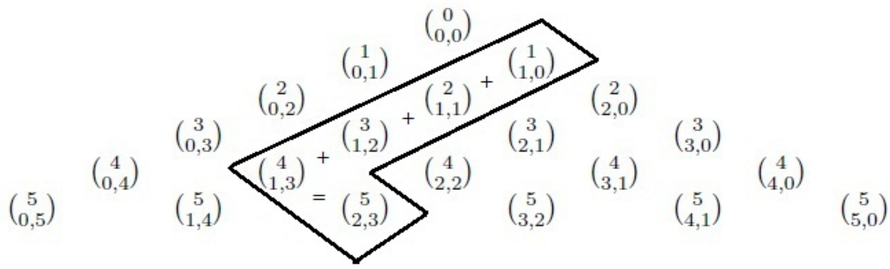

Figure 3: Illustration of (3.2)

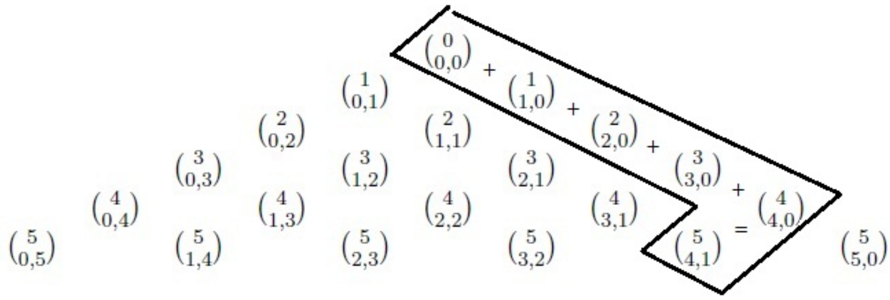

Figure 4: Illustration of (3.1)

4 Generating Functions

Let $(a_r) = (a_0, a_1, \ldots, a_r, \ldots)$ be a sequence of numbers. The *(ordinary) generating function* for the sequence (a_r) is defined to be the power series

$$A(x) = \sum_{r\geq 0} a_r x^r. \tag{4.1}$$

For instance, the generating function for the sequence

$$\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, \ldots\right),$$

where n is a non-negative integer is $(1+x)^n$. This is obtained throught the use of the Binomial theorem which is the case $y = 1$ in (1.2) given by

$$(1+x)^n = \sum_{r\geq 0} \binom{n}{r} x^r.$$

Note that this coincides with (1.3). On the otherhand, the *exponential generating function* for the sequence (a_r) is defined to be the power series

$$a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \ldots + a_r\frac{x^r}{r!} + \ldots = \sum_{r\geq 0} a_r\frac{x^r}{r!}. \tag{4.2}$$

Now, combining (1.1) and (1.3), we have

$$(1+x)^n = \sum_{r\geq 0} \frac{(n)_r}{r!} x^r.$$

This means that $(1+x)^n$ is the exponential generating function for the sequence

$$((n)_0, (n)_1, (n)_2, \ldots, (n)_n, 0, 0, \ldots).$$

Theorem 4.1. *The exponential generating function for the sequence* (a_r)*, where*

$$a_r = \sum_{0\leq r_1, r_2, \ldots, r_n \leq r} \binom{r}{r_1, r_2, \ldots, r_n}$$

is a sum taken over all n*-ary sequences* $(r_1, r_2, \ldots, r_n)$ *of non-negative integers with* $\sum_{j=1}^{n} r_j = r$*, is*

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right)^n = e^{xn}. \tag{4.3}$$

Remark 4.1. It is easy to verify that when $n = 1$ in (4.3), we get the exponential generating function of the sequence $(1, 1, 1, \ldots, 1, \ldots)$ which is e^x.

Proof of Theorem 4.1. Clearly, we have

$$\begin{aligned} e^{xn} &= e^x \cdot e^x \cdot \ldots \cdot e^x \\ &= \left(\sum_{r_1\geq 0} \frac{x^{r_1}}{r_1!}\right) \left(\sum_{r_2\geq 0} \frac{x^{r_2}}{r_2!}\right) \cdots \left(\sum_{r_n\geq 0} \frac{x^{r_n}}{r_n!}\right) \\ &= \sum_{r\geq 0} \left\{ \sum_{0\leq r_1, r_2, \ldots, r_n \leq r} \frac{x^{r_1}}{r_1!} \cdot \frac{x^{r_2}}{r_2!} \frac{x^{r_3}}{r_3!} \cdots \cdot \frac{x^{r_n}}{r_n!} \right\} \\ &= \sum_{r\geq 0} \left\{ \sum_{0\leq r_1, r_2, \ldots, r_n \leq r} \left(\frac{1}{r_1!} \cdot \frac{1}{r_2!} \frac{1}{r_3!} \cdots \cdot \frac{1}{r_n!}\right) x^r \right\}, \end{aligned}$$

where the sum is taken over all n-ary sequences $(r_1, r_2, \ldots, r_n)$ of non-negative integers, where $\sum_{j=1}^{n} r_j = r$. Now,

$$\begin{aligned} e^{xn} &= \sum_{r\geq 0} \left\{ \sum_{0\leq r_1, r_2, \ldots, r_n \leq r} \frac{r!}{r_1!, r_2!, \ldots, r_n!} \frac{x^r}{r!} \right\} \\ &= \sum_{r\geq 0} \left\{ \sum_{0\leq r_1, r_2, \ldots, r_n \leq r} \binom{r}{r_1, r_2, \ldots, r_n} \frac{x^r}{r!} \right\}. \end{aligned}$$

Thus, the proof is done. □

Remark 4.2. We can also prove Theorem 4.1 using the identity in (1.15). That is, by multiplying both sides of (1.15) with $\frac{x^n}{n!}$ and summing up to infinity yields

$$\sum_{n\geq 0} \frac{(mx)^n}{n!} = \sum_{n\geq 0} \left\{ \sum_{0\leq n_1, n_2, \ldots, n_m \leq n} \binom{n}{n_1, n_2, \ldots, n_m} \frac{x^n}{n!} \right\}.$$

This is precisely the result in Theorem 4.1.

The next theorem is deduced from Theorem 4.1.

Theorem 4.2. *The ordinary generating function for the sequence (a_r), where*

$$a_r = \sum_{0\le r_1,r_2,\ldots,r_n\le r} \frac{1}{r_1!r_2!\ldots r_n!}$$

is a sum taken over all n-ary sequences $(r_1, r_2, \ldots, r_n)$ of non-negative integers with $\sum_{j=1}^{n} r_j = r$, is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\ldots\right)^n = e^{xn}. \tag{4.4}$$

5 Conclusion

In this study, we have obtained generalizations for some classical identities involving the Binomial coefficients via Multinomial coefficients. The identity obtained in Theorem 2.1 is a generalization of the known Vandermonde's identity since the latter is a particular case when the integer $k = 2$ in equation (2.2) of Theorem 2.1. Also, the Chu Shih-Chieh's identities in (1.8) and (1.9) appears to be particular cases of the results stated in Theorem 3.2 and Theorem 3.1, respectively. Some generating functions of sequences involving the Multinomial coefficients were also investigated and presented.

6 Recommendations

The authors recommend the following for further research:

1. The binomial coefficient satisfies the *orthogonality relation*

$$\sum_{j=i}^{n}(-1)^{n-j}\binom{n}{j}\binom{j}{i} = \sum_{j=i}^{n}(-1)^{j-i}\binom{n}{j}\binom{j}{i} = \delta_{ni}, \tag{6.1}$$

where $\delta_{ni} = \begin{cases} 0, n \neq i \\ 1, n = i \end{cases}$ is called *kronecker delta*. (6.1) can be obtained using the generating function in (1.3). Making use of (6.1), the *inverse relation*

$$f_n = \sum_{r=0}^{n}\binom{n}{r}g_r \Leftrightarrow g_n = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}f_r, \tag{6.2}$$

can be obtained. It would be compelling to establish the orthogonality and inverse relations for the multinomial coefficients.

2. A q-analogue of the multinomial coefficient is often defined as

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_m \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q![n_2]_q!\ldots[n_m]_q!}, \tag{6.3}$$

where $[n]_q! = \prod_{i=1}^{n}[i]_q$ is the q-*factorial of* n, $[n]_q = \frac{q^n-1}{q-1}$ is the q-*integer* n,

$$\begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_q = \begin{bmatrix} n \\ n_1 \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q![n-n_1]_q!} \tag{6.4}$$

is the q-*binomial coefficients* and

$$\lim_{q\to 1}\begin{bmatrix} n \\ n_1, n_2, \ldots, n_m \end{bmatrix}_q = \binom{n}{n_1, n_2, \ldots, n_m}.$$

(6.3) is called as q-*multinomial coefficients* (see Vinroot [15] and Warnaar [9]). It is known that the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ satisfy the q-binomial inversion formula

$$f_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Leftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k \tag{6.5}$$

seen in [2] and the q-*binomial Vandermonde convolution*

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{r=0}^{k} q^{r(m-k+r)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ k-r \end{bmatrix}_q \tag{6.6}$$

which was introduced by Bender [5] and further studied by Evans [6] and Sulanke [7]. One may investigate the possibility of establishing a q-multinomial version of (6.5) and (6.6) as well as the q-analogues of (3.1) and (3.2).

3. Corcino [13] defined the p, q-*binomial coefficients* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}, \tag{6.7}$$

where $p \neq q$, and obtained its fundamental properties. It is easy to verify that (6.7) satisfies

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \frac{[n]_{pq}!}{[k]_{pq}![n-k]_{pq}!}, \tag{6.8}$$

where $[n]_{pq} = \frac{p^n - q^n}{p-q}$ and $[n]_{pq}! = \prod_{j=1}^{n} [j]_{pq}$. Moreover, Lundow and Rosengren [14] used the p, q-binomial coefficients in (6.7) to describe the magnetization distribution of the *Ising Model*. It would be compelling to define a p, q-analogue of the multinomial coefficients and study its properties and possible applications.

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Competing Interests

The authors declare that no competing interests exist.

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