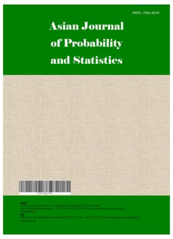

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Backward Doubly SDEs with weak Monotonicity and General Growth Generators

B. Mansouri[1*] and M. A. Saouli[1]

[1] *University Mohamed Khider, BP 145, 07000 Biskra, Algeria.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

We deal with backward doubly stochastic differential equations (BDSDEs) with a weak monotonicity and general growth generators and a square integrable terminal datum. We show the existence and uniqueness of solutions. As application, we establish the existence and uniqueness of Sobolev solutions to some semilinear stochastic partial differential equations (SPDEs) with a general growth and a weak monotonicity generators. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

Keywords: Backward doubly stochastic differential equations; weak monotonicity; Sobolev solutions; semilinear stochastic partial differential equations.

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**Corresponding author: E-mail: mansouri.badreddine@gmail.com, b.mansouri@univ-biskra.dz;*

1 Introduction

Backward doubly stochastic differential equation (BDSDE for short) at the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s - \int_t^T Z_s dW_s\ , \qquad 0 \le t \le T \qquad \left(E^{\xi,f,g}\right)$$

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t.

The existence and uniqueness of solutions to BDSDEs of type $(E^{\xi,f,g})$ were first established in Pardoux and Peng [1] they have proved the existence and uniqueness under uniformly Lipschitz conditions. Since then, the BDSDEs have been intensively studied and a lot of papers were devoted to the development of the theory of BDSDEs as well as their relation with the stochastic optimal control problems see [2], [3], [4], and stochastic partial differential equations(SPDEs), we are especially concerned in this paper with the last connection. Was firstly initiated by Pardoux and Peng [1] to give probabilistic interpretation for the solutions of a class of semilinear SPDEs where the coefficients are smooth enough, the idea is to connect the following BDSDEs system

$$\begin{aligned} Y_s^{t,x} &= h\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_s^T g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) d\overleftarrow{B}_r - \int_s^T Z_r^{t,x} dW_r, \\ X_s^{t,x} &= x + \int_t^s b\left(X_r^{t,x}\right) dr + \int_t^s \sigma\left(X_r^{t,x}\right) dW_r, \end{aligned}$$

with the following semilinear SPDE

$$\begin{aligned} u(t,x) = h(x) &+ \int_s^T \big(\mathcal{L}u(r,x) + f(r,x,u(r,x), \sigma^* \nabla u(r,x))\big) dr \\ &+ \int_s^T g(r,x,u(r,x), \sigma^* \nabla u(r,x))\, d\overleftarrow{B}_r, \quad t \le s \le T, \end{aligned}$$

where

$$\mathcal{L} := \frac{1}{2} \sum_{i,j} (a_{ij}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad \text{with } (a_{ij}) := \sigma\sigma^*.$$

After what's realised by Pardoux and Peng [1] numerous authors show the connections between BDSDEs and solutions of stochastic partial differential equations. Bally and Matoussi [5], and [6], [7], studied the solutions of quasilinear SPDEs in Sobolev spaces in terms of BDSDEs with Lipschitz coefficients, in Bahlali et all [8], and [9], they have prove the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDEs) with superlinear growth gernerator, Zhang and Zhao [10] considered BDSDEs under Lipschitz conditions in spatial integral form on infinite horizon and related their solutions with the stationary solutions of certain SPDEs. And then [11], [12], [13] studied the same BDSDE but under linear growth and monotonicity conditions. They also proved that the solution of finite horizon BDSDE gives the solution of the initial value problem of the corresponding SPDE in Sobolev space see [14], and the solution of the infinite horizon BDSDE gives the stationary solution of the SPDEs.

Due to the application of BDSDEs, many works have been made to relax the assumptions on the driver f. see for example [8], [15], [16], [17], [18], where Shi et al [16], and [19], [20], provided a comparison theorem which is very important in studying viscosity solution of SPDEs with stochastic tools, and Bahlali et al [8] provided the existence and uniqueness in the case with a superlinear growth generator and a square integrable terminal datum.

In this paper, we obtain existence and uniqueness results for BDSDEs when the coefficient f has a weak monotonicity and general growth in y and lipschitz in the variable z, secondly we connect this kind of BDSDEs with the corresponding semilinear SPDEs with superlinear generator for which we establish the existence and uniqueness of Sobolev solutions. The rest of paper is organized as follows.

- In Section 2, we will present some preliminary notations needed in the whole paper.
- In Section 3, we give the estimate for the solutions of BDSDE $\left(E^{\xi,f,g}\right)$.
- In Section 4, we consider our main results, the existence and uniqueness of solution for BDSDE $\left(E^{\xi,f,g}\right)$.
- In Section 5, we give an application to Sobolev solutions for semilinear SPDEs.

2 Notations, Assumptions and Definitions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T>0$, let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. We put, $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub $\sigma-$ fields, and hence it is not a filtration.

For each $t \in [0,T]$, we define

$$\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B,$$

the collection $(\mathcal{G}_t)_{t\in[0,T]}$ is a filtration.

For any d, $k \geq 1$, we consider the following spaces of processus:

- Let $\mathcal{M}^2\left(0,T,\mathbb{R}^d\right)$ denote the set of $d-$dimensional, $\mathcal{F}_t-$measurable stochastic processes $\{\varphi_t; t\in[0,T]\}$, such that $E\int_0^T \left|\varphi_t\right|^2 dt < \infty$.
- We denote by $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right)$ the set of $k-$dimensional continuous, $\mathcal{F}_t-$ measurable stochastic processes $\{\varphi_t; t\in[0,T]\}$, which satisfy $E\big(\sup_{0\leq t\leq T}\left|\varphi_t\right|^2\big) < \infty$.
- $\mathbb{L}^2$ the set of $\mathcal{F}_T$- measurable random variables $\xi : \Omega \to \mathbb{R}^k$ with $\mathbb{E}|\xi|^2 < +\infty$.

Let $f : \Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \longmapsto \mathbb{R}^k$, $g : \Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \longmapsto \mathbb{R}^{k\times l}$ be measurable functions such that, for every $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $f(\cdot,y,z)\in\mathcal{M}^2\left(0,T,\mathbb{R}^k\right)$ and $g(\cdot,y,z)\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times l}\right)$.

Now, we consider the following BDSDE

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,ds + \int_t^T g(s,Y_s,Z_s)\,d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \qquad 0\leq t\leq T \qquad (E^{\xi,f,g}),$$

ξ is called the terminal datum and f the generator.

Definition 2.1. *A solution of equation* $\left(E^{\xi,f,g}\right)$ *is a couple* (Y,Z) *which belongs to the space* $\mathcal{S}^2\left([0,T],\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ *and satisfies* $\left(E^{\xi,f,g}\right)$.

We consider the following assumptions:

(H.1) $dP\times dt$-a.e. (almost everywhere), $z\in\mathbb{R}^{k\times d}$, $y\to f(w,t,y,z)$ is continuous.

(H.2) f satisfies the weak monotonicity condition in y, i.e., there exist a nondecreasing and concave function $k(\cdot):\mathbb{R}^+\to\mathbb{R}^+$, with $k(u)>0$ for $u>0$, $k(0)=0$ and $\int_{0^+}k^{-1}(u)du=+\infty$ such that $dP\times dt$-a.e.,$\forall\,(y_1,y_2)\in\mathbb{R}^{2k}$, $z\in\mathbb{R}^{k\times d}$,

$$\langle y_1-y_2, f(t,\omega,y_1,z)-f(t,\omega,y_2,z)\rangle\leq k\left(\left|y_1-y_2\right|^2\right).$$

(H.3) i) f is Lipschitz in z, uniformly with respect to (w,t,y) i.e., there exists a constant $c>0$ such that $\forall y\in\mathbb{R}^k$, and $\forall z,z'\in\mathbb{R}^{k\times d}$, $dP\times dt$-a.e.,

$$\left|f(w,t,y,z)-f\left(w,t,y,z'\right)\right|\leq c\left|z-z'\right|.$$

ii) There exists a constant $c>0$ and a constant $0<\alpha\leq\frac{1}{4}$ such that $dP\times dt$-a.e.,

$$\left|g(w,t,y,z)-g\left(w,t,y^{'},z'\right)\right|\leq c\left|y-y^{'}\right|+\alpha\left|z-z'\right|.$$

(H.4) f for y has a general growth, i.e., $dP\times dt$-a.e., $\forall y\in\mathbb{R}^k$

$$\left|f(t,\omega,y,0)\right|\leq\left|f(t,\omega,0,0)\right|+\varphi\left(\left|y\right|\right),$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is increasing continuous function.

(H.5)

$$\begin{cases} f(t,\omega,0,0)\in\mathcal{M}^2\left(0,T,\mathbb{R}^k\right), \\ g(t,\omega,0,0)\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times l}\right). \end{cases}$$

3 Estimate for the solutions of BDSDE $\left(E^{\xi,f,g}\right)$.

We propose the following assumption on f and g.

(H.6) $dP\times dt$-a.e., $\forall(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$

$$\langle y, f(t,\omega,y,z)\rangle\leq\psi\left(\left|y\right|^2\right)+\lambda\left|y\right|\left|z\right|+\left|y\right|\sigma_t,$$

where λ is a positive constant, σ_t is a positive and $(\mathcal{F}_t)$ progressively measurable processus with $E\int_0^T\left|\sigma_t\right|^2dt<\infty$ and $\psi(\cdot)$ is a nondecreasing concave function from $\mathbb{R}^+$ to itself with $\psi(0)=0$.

(H.7) $dP\times dt$-a.e., $\forall(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$

$$\left|g(t,\omega,y,z)\right|^2\leq\lambda\left|y\right|^2+\gamma\left|z\right|^2+\eta_t,$$

with λ is a positive constant, $0<\gamma\leq\frac{1}{4}$ and η_t is a positive and $(\mathcal{F}_t)$ progressively measurable processus with $E\int_0^T\eta_t dt<\infty$.

Proposition 3.1. *Let f and g satisfy $(H.6)$ and $(H.7)$, let $(Y_t,Z_t)_{t\in[0,T]}$ be a solution to the BDSDE with parameters (ξ,T,f,g). Then for each $\delta>0$ there exists a constants $K>0$ depending only on δ,λ and γ such that*

(i) *for each $0\leq t\leq T$:*

$$\begin{aligned}\mathbb{E}(\sup_{0\leq s\leq T}|Y_s|^2)+\mathbb{E}\left(\int_t^T|Z_s|^2ds\right) \quad\leq\quad & \left(\mathbb{E}|\xi|^2+2\int_t^T\psi\left(\mathbb{E}|Y_s|^2\right)ds+\frac{1}{\delta}\mathbb{E}\int_t^T|\sigma_s|^2ds\right.\\ & \left.+\mathbb{E}\int_t^T\eta_s ds\right)K\exp\left(K(T-t)\right).\end{aligned}$$

(ii) *Moreover for each* $\delta > 0$ *there exists a constants* $\bar{K} > 0$ *depending only on* δ, λ *and* γ *such that for* $0 \leq r \leq t \leq T$:

$$\begin{aligned} \mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\Big|\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2ds\Big|\mathcal{F}_r\right) &\leq \Big(\mathbb{E}\left(|\xi|^2\Big|\mathcal{F}_r\right)+2\int_t^T\psi\left(\mathbb{E}\left(|Y_s|^2\Big|\mathcal{F}_r\right)\right)ds \\ &+\frac{1}{\delta}\mathbb{E}\left(\int_t^T|\sigma_s|^2\,ds\Big|\mathcal{F}_r\right)+2\mathbb{E}\left(\int_t^T\eta_s ds\Big|\mathcal{F}_r\right)\Big)\bar{K}\exp\left(\bar{K}T\right). \end{aligned}$$

Proof: For the first part, applying It's formula to $|Y_t|^2$ yields that, for each $0 \leq t \leq T$,

$$\begin{aligned} |Y_t|^2+\int_t^T|Z_s|^2ds &= |\xi|^2+2\int_t^T\langle Y_s,f\left(s,Y_s,Z_s\right)\rangle ds+2\int_t^T\langle Y_s,g\left(s,Y_s,Z_s\right)\rangle dB_s \\ &\quad -2\int_t^T\langle Y_s,Z_s\rangle dW_s+\int_t^T|g(s,Y_s,Z_s)|^2ds, \end{aligned}$$

taking expectation, we get

$$\begin{aligned} \mathbb{E}\left|Y_t\right|^2+\mathbb{E}\int_t^T\left|Z_s\right|^2ds &= \mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_t^T\langle Y_s,f\left(s,Y_s,Z_s\right)\rangle ds+2\mathbb{E}\int_t^T\langle Y_s,g\left(s,Y_s,Z_s\right)\rangle dB_s \\ &\quad -2\mathbb{E}\int_t^T\langle Y_s,Z_s\rangle dW_s+\mathbb{E}\int_t^T\left|g(s,Y_s,Z_s)\right|^2ds. \end{aligned}$$

Now, by $(H.6)$ and Young's inequality, we have

$$\begin{aligned} 2\int_t^T\langle Y_s,f(s,Y_s,Z_s)\rangle ds &\leq 2\int_t^T\Big(\psi(|Y_s|^2)+\lambda\left|Y_s\right|\left|Z_s\right|+\left|Y_s\right|\sigma_s\Big)ds, \\ &\leq 2\int_t^T\psi\left(|Y_s|^2\right)ds+\left(2\lambda^2+\delta\right)\int_t^T|Y_s|^2\,ds \\ &\quad +\int_t^T\frac{|\sigma_s|^2}{\delta}ds+\int_t^T\frac{|Z_s|^2}{2}ds, \end{aligned}$$

Then by $(H.7)$, we have

$$\begin{aligned} \mathbb{E}\left|Y_t\right|^2+\left(\frac{1}{2}-\gamma\right)\mathbb{E}\int_t^T\left|Z_s\right|^2ds &\leq \mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_t^T\psi\left(|Y_s|^2\right)ds+\left(2\lambda^2+\lambda+\delta\right)\mathbb{E}\int_t^T|Y_s|^2\,ds \\ &\quad +\frac{1}{\delta}\mathbb{E}\int_t^T|\sigma_s|^2\,ds+\mathbb{E}\int_t^T\eta_s ds. \end{aligned}$$

Since $\int_0^t\langle Y_s,Z_s\rangle dW_s$ and $\int_0^t\langle Y_s,g(s,Y_s,Z_s)\rangle dB_s$ are a uniformly integrable martingale. For each $0 \leq t \leq T$, we have the following inequality

$$\left(\frac{1}{2}-\gamma\right)\mathbb{E}\int_t^T|Z_s|^2\,ds\leq\Delta_t, \tag{3.1}$$

where,

$$\Delta_t=\mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_t^T\psi(|Y_s|^2)ds+\left(2\lambda^2+\lambda+\delta\right)\mathbb{E}\int_t^T|Y_s|^2\,ds+\frac{1}{\delta}\mathbb{E}\int_t^T|\sigma_s|^2\,ds+\mathbb{E}\int_t^T\eta_s ds.$$

Furthermore, it follows from the Burkholder-Davis-Gundy and Young's inequality, we have

$$\begin{aligned} 2\mathbb{E}\left(\sup_{t\leq u\leq T}\left|\int_u^T\langle Y_s,Z_s\rangle dW_s\right|\right) &\leq 2C_p\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|\sqrt{\int_t^T|Z_s|^2\,ds}\right), \\ &\leq \frac{1+2\gamma}{2}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\right)+\frac{2C_p^2}{1+2\gamma}\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\right), \quad (3.2) \\ &< \infty. \end{aligned}$$

By assumptions $(H.6)$, $(H.7)$ and using $(3.1)-(3.2)$, we have

$$\mathbb{E}\left(\sup_{t\leq s\leq T}|Y_s|^2\right)+\mathbb{E}\int_t^T|Z_s|^2\,ds\leq\left(\frac{2}{1-2\gamma}\right)\left(1+\frac{4C_p^2}{(1+2\gamma)(1-2\gamma)}\right)\mathbb{E}\left(\Delta_t\right).$$

Jensen's inequality, Gronwall's Lemma and Fubini's theorem, in view of the concavity condition of $\psi(\cdot)$, then there exists a constant $K>0$ such that $t\in[0,T]$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s|^2\right)+\mathbb{E}\int_t^T|Z_s|^2ds \leq & \Big(K\mathbb{E}|\xi|^2+2K\int_t^T\psi(\mathbb{E}|Y_s|^2)ds+\frac{K}{\delta}\mathbb{E}\int_t^T|\sigma_s|^2ds\\ &+K\mathbb{E}\int_t^T\eta_s ds\Big)\exp\left(K\left(T-t\right)\right).\end{aligned}$$

For the second part, we will use the same operation but applied the conditional expectation with respect to $\mathcal{F}_r, r\in[t,T]$ instead of the mathematical expectation.

Using the Burkholder-Davis-Gundy, $2ab\leq\frac{a^2}{\epsilon}+\epsilon b^2$ inequalities and assumption $(H.7)$, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq u\leq T}\left|\int_u^T\langle Y_s,g(s,Y_s,Z_s)\rangle dB_s\right|\,\middle|\,\mathcal{F}_r\right) &\leq C_p\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|\sqrt{\int_t^T|g(s,Y_s,Z_s)|^2\,ds}\,\middle|\,\mathcal{F}_r\right),\\ &\leq \frac{1}{2\epsilon}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)+\frac{\epsilon C_p^2}{2}\mathbb{E}\left(\int_t^T|g(s,Y_s,Z_s)|^2\,ds\,\middle|\,\mathcal{F}_r\right),\\ &\leq \left(\frac{1}{2\epsilon}+\frac{\epsilon\lambda C_p^2}{2}\right)\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)+\frac{\epsilon\gamma C_p^2}{2}\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right)\\ &+ \frac{\epsilon C_p^2}{2}\mathbb{E}\left(\int_t^T\eta_s ds\,\middle|\,\mathcal{F}_r\right)\\ &< \infty. \end{aligned}\tag{3.3}$$

Applying It's formula to $|Y_t|^2, \forall t\in[0,T]$, and we using $(H.6)$, $(H.7)$, (3.3) and $\mathbb{E}\left(\int_t^T\langle Y_s,Z_s\rangle dW_s\,\middle|\,\mathcal{F}_r\right)=0$, we have for any $0\leq r\leq t\leq T$

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right) &\leq \mathbb{E}\left(\left(\Delta_t\right)|\,\mathcal{F}_r\right)+\left(\frac{1}{2\epsilon}+\frac{\epsilon\lambda C_p^2}{2}\right)\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)\\ &+ \left(\frac{1}{2}+\left(\frac{2+\epsilon C_p^2}{2}\right)\gamma\right)\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right)\\ &+ \frac{\epsilon C_p^2}{2}\mathbb{E}\left(\int_t^T\eta_s ds\,\middle|\,\mathcal{F}_r\right).\end{aligned}$$

Since $0\leq\gamma\leq\frac{1}{4}$ it is enough to take $C_p^2=\frac{1}{\epsilon^2}$, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2ds\,\middle|\,\mathcal{F}_r\right) &\leq \mathbb{E}\left(\left(\Delta_t\right)|\,\mathcal{F}_r\right)+\frac{1+\lambda}{2\epsilon}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)\\ &+ \frac{6\epsilon+1}{8\epsilon}\mathbb{E}\left(\int_t^T|Z_s|^2ds\,\middle|\,\mathcal{F}_r\right)+\frac{1}{2\epsilon}\mathbb{E}\left(\int_t^T\eta_s ds\,\middle|\,\mathcal{F}_r\right),\end{aligned}$$

we choosing $\epsilon=\frac{3+4\lambda}{6}$, get

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right) &\leq \mathbb{E}\left(\left(\Delta_t\right)|\,\mathcal{F}_r\right)+\frac{3\left(\lambda+1\right)}{4\lambda+3}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\,\middle|\,\mathcal{F}_r\right)\\ &+ \frac{3\left(\lambda+1\right)}{4\lambda+3}\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right)+\frac{3}{4\lambda+3}\mathbb{E}\left(\int_t^T\eta_s ds\,\middle|\,\mathcal{F}_r\right),\end{aligned}$$

since $0<\frac{3(\lambda+1)}{4\lambda+3}<1$, we obtain

$$\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2+\int_t^T|Z_s|^2\,ds\,\middle|\,\mathcal{F}_r\right)\leq\frac{4\lambda+3}{\lambda}\left(\mathbb{E}\left(\left(\Delta_t\right)|\,\mathcal{F}_r\right)+\frac{3}{4\lambda+3}\mathbb{E}\left(\int_t^T\eta_s ds\,\middle|\,\mathcal{F}_r\right)\right),$$

from which together with Gronwall's Lemma, Fubini's theorem and Jensen's inequality, in view of the concavity condition of $\psi(\cdot)$ then there exists a constants $\bar{K} > 0$ such that for $0 \leq r \leq t \leq T$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq u\leq T}|Y_u|^2\Big|\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2\,ds\Big|\mathcal{F}_r\right) \quad\leq\quad & \left(\mathbb{E}\left(|\xi|^2\Big|\mathcal{F}_r\right)+2\int_t^T\psi\left(\mathbb{E}\left(|Y_s|^2\Big|\mathcal{F}_r\right)\right)ds+\frac{1}{\delta}\mathbb{E}\left(\int_t^T|\sigma_s|\,ds\Big|\mathcal{F}_r\right)\right.\\ & \left.+2\mathbb{E}\left(\int_t^T\eta_s ds\Big|\mathcal{F}_r\right)\right)\bar{K}\exp\left(\bar{K}T\right),\end{aligned}$$

hence the required result. ■

4 The Main Results

Theorem 4.1. *Let $\xi \in \mathbb{L}^2$, assume that **(H.1)–(H.5)** are satisfied. Then equation $\left(E^{f,g,\xi}\right)$ has a unique solution.*

4.1 Proof of uniqueness

Suppose that f and g satisfies the assumption $(H.1)-(H.5)$. Let $\left(Y_t^1, Z_t^1\right)$ and $\left(Y_t^2, Z_t^2\right)$ be two solutions of the BDSDE with parameters (ξ, T, f, g). Then $\left(\bar{Y}_t, \bar{Z}_t\right) = \left(Y_t^1 - Y_t^2, Z_t^1 - Z_t^2\right)$ is a solution to the following BDSDE

$$\bar{Y}_t = \int_t^T \bar{f}\left(s, \bar{Y}_s, \bar{Z}_s\right) ds + \int_t^T \bar{g}\left(s, \bar{Y}_s, \bar{Z}_s\right) dB_s - \int_t^T \bar{Z}_s dW_s, \qquad t \in [0, T],$$

where for each $\left(\bar{Y}_t, \bar{Z}_t\right) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\begin{cases} \bar{f}\left(t, \bar{Y}_t, \bar{Z}_t\right) = f\left(t, \bar{Y}_t + Y_t^2, \bar{Z}_t + Z_t^2\right) - f\left(t, Y_t^2, Z_t^2\right), \\ \bar{g}\left(t, \bar{Y}_t, \bar{Z}_t\right) = g\left(t, \bar{Y}_t + Y_t^2, \bar{Z}_t + Z_t^2\right) - g\left(t, Y_t^2, Z_t^2\right). \end{cases}$$

It follows from $(H.2)$ and $(H.3)\,(i)$ that $dP \times dt - a.e.$,

$$\begin{aligned}\langle\bar{Y}, \bar{f}\left(t, \bar{Y}, \bar{Z}\right)\rangle &= \langle\bar{Y}, f\left(t, \bar{Y} + Y^2, \bar{Z} + Z^2\right) - f\left(t, Y^2, Z^2\right)\rangle \\ &\leq k\left(\left|\bar{Y}\right|^2\right) + c\left|\bar{Y}\right|\left|\bar{Z}\right|,\end{aligned}$$

then the generator $\bar{f}\left(s, \bar{Y}_s, \bar{Z}_s\right)$ of BDSDE with $\psi(u) = k(u)$, $\lambda = c$, $\sigma_t = 0$ satisfied the assumption $(H.6)$.

It follows from $(H.3)\,(ii)$ that $dP \times dt - a.e.$,

$$\left|\bar{g}\left(t, \bar{Y}, \bar{Z}\right)\right|^2 \leq c\left|\bar{Y}\right|^2 + \alpha\left|\bar{Z}\right|^2,$$

then the assumption $(H.7)$ is satisfied for the generator $\bar{g}\left(s, \bar{Y}_s, \bar{Z}_s\right)$ of BDSDE with $\alpha = \gamma$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$, we have

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\bar{Y}_s\right|^2\right) + \mathbb{E}\left(\int_t^T\left|\bar{Z}_s\right|^2 ds\right) \leq C\int_t^T\left(k\left(\mathbb{E}\sup_{s\leq u\leq T}\left|\bar{Y}_u\right|^2\right)\right)ds,$$

where $C = 2K\exp(KT)$ in view of $\int_{0^+} k^{-1}(u)\,du = \infty$, Bihari's inequality yields that, $\forall t \in [0, T]$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\bar{Y}_s\right|^2 + \int_t^T\left|\bar{Z}_s\right|^2 ds\right) = 0.$$

The proof of the uniqueness is then complete. ■

4.2 Proof of existence

Let ϕ be a function of $C^{\infty}\left(\mathbb{R}^k,\mathbb{R}^+\right)$ with the closed unit as compact support, and satisfies $\int_{\mathbb{R}^k}\phi(v)\,dv = 1$. For each $n\geq 1$ and each $(\omega,t,Y)\in\Omega\times[0,T]\times\mathbb{R}^k$, we set

$$\begin{aligned} f_n(t,Y_t,V_t) &= n^k f(t,Y_t,V_t)*\phi(nY_t),\\ &= n^k\int_{\mathbb{R}^k} f(t,v,V_t)\,\phi(n(Y_t-v))\,dv. \end{aligned} \tag{4.1}$$

Then f_n is an $(\mathcal{F}_t)$-progressively measurable process for each $Y\in\mathbb{R}^k$ and

$$\begin{aligned} f_n(t,Y_t,V_t) &= \int_{\mathbb{R}^k} f\left(t,Y_t-\frac{v}{n},V_t\right)\phi(v)\,dv,\\ &= \int_{\{v:|v|\leq 1\}} f\left(t,Y_t-\frac{v}{n},V_t\right)\phi(v)\,dv. \end{aligned} \tag{4.2}$$

Let us turn to the existence part. The proof will be split into three lemmas and after the proof of Theorem 4.1.

Lemma 4.1. *Let f and g satisfies the hypothesis **(H.1)–(H.5)**, $V\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ and $\xi\in\mathbb{L}^2\left(\mathcal{F}_T,\mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP-a.s., \left|\xi\right|\leq\beta \quad dP\times dt-a.e., \left|g(t,\omega,0,0)\right|\leq\beta \quad \left|f(t,\omega,0,0)\right|\leq\beta \quad and \quad \left|V_t\right|\leq\beta. \tag{4.3}$$

Then there exists a unique solution to the following BDSDE:

$$Y_t=\xi+\int_t^T f(s,Y_s,V_s)\,ds+\int_t^T g(s,Y_s,V_s)\,d\overleftarrow{B}_s-\int_t^T Z_s dW_s \quad t\in[0,T]. \tag{4.4}$$

Proof: It follows from $(H.3)\,(i)$, $(H.4)$ and (4.3) that, for each $Y\in\mathbb{R}^k$ $dP\times dt-a.e.$,

$$\left|f(s,Y_s,V_s\right|\leq c\beta+\beta+\varphi(|Y_s|). \tag{4.5}$$

Thus, checked from (4.1) that for each $n\geq 1$, $f_n(t,Y_t,V_t)$ is locally Lipschitz in Y uniformly with respect to (t,ω). Furthermore, for each $n\geq 1$ and $Y\in\mathbb{R}^k$, it follows from (4.2) and (4.5) that $dP\times dt-a.e.$,

$$\begin{aligned} \left|f_n(t,Y_t,V_t)\right| &= \Big|\int_{\{v:|v|\leq 1\}} f(t,Y_t-\frac{v}{n},V_t)\phi(v)\,dv\Big|,\\ &\leq (c\beta+\beta+\varphi(|Y_t|+1))\int_{\{v:|v|\leq 1\}}\phi(v)\,dv = c\beta+\beta+\varphi(|Y_t|+1). \end{aligned} \tag{4.6}$$

Now, for some large enough integer $u>0$ which will be chosen later, let ρ_u be a smooth function such that $0\leq\rho_u\leq 1$, $\rho_u(Y_t)=1$ for $|Y_t|\leq u$ and $\rho_u(Y_t)=0$ as soon as $|Y_t|\geq u+1$. Then for each $n\geq 1$, the function $\rho_u(Y_t)\,f_n(t,Y_t,V_t)$ is Lipschitz in Y, uniformly with respect to (t,ω).

Thus, from Pardoux-Peng [15], we know that for each $n\geq 1$, the following BDSDE has a unique solution $(Y_t^n,Z_t^n)_{t\in[0,T]}$:

$$Y_t^n=\xi+\int_t^T\rho_u(Y_s^n)\,f_n(s,Y_s^n,V_s)\,ds+\int_t^T g(s,Y_s^n,V_s)\,d\overleftarrow{B}_s-\int_t^T Z_s^n dW_s, \quad 0\leq t\leq T. \tag{4.7}$$

It follows from $(H.2)$ and (4.2) that for each $n\geq 1$ and $\left(Y_t^1,Y_t^2\right)\in\mathbb{R}^{2k}$, $dP\times dt-a.e.$,

$$\langle Y_t^1-Y_t^2, f_n\left(t,Y_t^1,V_t\right)-f_n\left(t,Y_t^2,V_t\right)\rangle\leq\int_{\{v:|v|\leq 1\}} k\left(\left|Y_t^1-Y_t^2\right|^2\right)\phi(v)\,dv = k\left(\left|Y_t^1-Y_t^2\right|^2\right). \tag{4.8}$$

For each $n \geq 1$ and $Y_t \in \mathbb{R}^k$, combing (4.6) and (4.8) yields that $dP \times dt - a.e.$,

$$\begin{aligned} \langle Y_t, \rho_u(Y_t) f_n(t, Y_t, V_t)\rangle &= \rho_u(Y_t)\langle Y_t, f_n(t, Y_t, V_t)\rangle, \\ &\leq k\left(|Y_t|^2\right) + |Y_t|\left(c\beta + \beta + \varphi(1)\right), \end{aligned}$$

Then the assumption $(H.6)$ is satisfied for the generator $\rho_u(Y_t^n) f_n(t, Y_t^n, V_t)$ of BDSDE (4.7) with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_t = c\beta + \beta + \varphi(1)$.

It follows from $(H.3)\,(ii)$ that $dP \times dt - a.e.$,

$$\begin{aligned} |g(t, Y_t^n, V_t)|^2 &\leq 2\,|g(t, Y_t^n, V_t) - g(t, 0, 0)|^2 + 2\,|g(t, 0, 0)|^2, \\ &\leq 2c\,|Y_t^n|^2 + 2\alpha^2\,|V_t|^2 + 2\,|g(t, 0, 0)|^2. \end{aligned}$$

Then the generator $g(t, Y_t^n, V_t)$ of BDSDE (3.10) with $\lambda = 2c$, $\gamma = 2\alpha^2$ and $\eta_t = 2|g(t, \omega, 0, 0)|^2$ satisfied the assumption $(H.7)$.

Thus, it follow from Proposition 3.1 (ii) with $\delta = 1$ that there exists a constant $\bar{K} > 0$ depending only on δ, λ and γ such that, for each $0 \leq r \leq t \leq T$, we have

$$\begin{aligned} \mathbb{E}\left(|Y_t^n|^2 \Big| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T |Z_s^n|^2\, ds \Big| \mathcal{F}_r\right) \leq \Big(&\mathbb{E}\left(|\xi|^2 \Big| \mathcal{F}_r\right) + 2\int_t^T k\left(\mathbb{E}\left(|Y_s^n|^2 \Big| \mathcal{F}_r\right)\right) ds + \left(c\beta + \beta + \varphi(1)\right)^2 T \\ &+ 4\mathbb{E}\left(\int_t^T |g(s, \omega, 0, 0)|^2\, ds \Big| \mathcal{F}_r\right)\Big)\bar{K}\exp\left(\bar{K}T\right). \end{aligned}$$

Note $\bar{\theta} = \bar{K}\exp\left(\bar{K}T\right)$ and using the (4.3), we get

$$\mathbb{E}\left(|Y_t^n|^2 \Big| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T |Z_s^n|^2\, ds \Big| \mathcal{F}_r\right) \leq \bar{\theta}\beta^2 + 2\bar{\theta}\int_t^T k\left(\mathbb{E}\left(|Y_s^n|^2 \Big| \mathcal{F}_r\right)\right) ds + \bar{\theta}\left(c\beta + \beta + \varphi(1)\right)^2 T + 4\bar{\theta}\beta^2 T.$$

Furthermore, since $k(\cdot)$ is a nondecreasing and concave function with $k(0) = 0$ it increases at most linearly, i.e., there exists $A > 0$ such that $k(x) \leq A(x + 1)$ for each $x \geq 0$, yields that

$$\begin{aligned} \mathbb{E}\left(|Y_t^n|^2 \Big| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T |Z_s^n|^2\, ds \Big| \mathcal{F}_r\right) &\leq \bar{\theta}\beta^2(4T + 1) + 2\bar{\theta}AT + \bar{\theta}\left(c\beta + \beta + \varphi(1)\right)^2 T \\ &\quad + 2\bar{\theta}A\int_t^T \mathbb{E}\left(|Y_s^n|^2 \Big| \mathcal{F}_r\right) ds. \end{aligned}$$

By Gronwall's lemma and with $r = t$, yields that

$$|Y_t^n|^2 + \mathbb{E}\left(\int_t^T |Z_s^n|^2\, ds\right) \leq u^2.$$

where $u^2 = \left(\bar{\theta}\beta^2(4T + 1) + 2A\bar{\theta}T + \bar{\theta}\left(c\beta + \beta + \varphi(1)\right)^2 T\right)\exp\left(2A\bar{\theta}T\right)$. By the previous inequality, yields that for each $n \geq 1$ and $\forall t \in [0, T]$

$$\begin{cases} |Y_t^n|^2 \leq u^2, \\ \mathbb{E}\left(\int_0^T |Z_s^n|^2\, ds\right) \leq u^2. \end{cases} \tag{4.9}$$

By (4.7) and (4.9), we can conclude that $(Y_t^n, Z_t^n)_{t\in[0,T]}$ solves the following BDSDE:

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, V_s)\, ds + \int_t^T g(s, Y_s^n, V_s)\, d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T. \tag{4.10}$$

In the sequel, we shall show that $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the space $\mathcal{S}^2\left(0, T, \mathbb{R}^k\right) \times \mathcal{M}^2\left(0, T, \mathbb{R}^{k\times d}\right)$.

In fact, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$ and $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$. Then for each $0 \leq t \leq T$

$$\Delta Y_t^{n,m} = \int_t^T \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) ds + \int_t^T \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) dB_s - \int_t^T \Delta Z_s^{n,m} dW_s, \quad (4.11)$$

where

$$\begin{cases} \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = f_n\left(s, \Delta Y_s^{n,m} + Y_s^m, V_s\right) - f_m\left(s, Y_s^m, V_s\right), \\ \\ \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = g\left(s, \Delta Y_s^{n,m} + Y_s^m, V_s\right) - g\left(s, Y_s^m, V_s\right). \end{cases}$$

It follows (4.8) that for each $\Delta Y_t^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.$,

$$\langle \Delta Y_t^{n,m}, \Delta f^{n,m}(t, \Delta Y_t^{n,m}, V_t)\rangle \leq k\left(|\Delta Y_t^{n,m}|^2\right) + |\Delta Y_t^{n,m}|\,|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)|.$$

Then the the generator $\Delta f^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ of BDSDE (4.11) with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_t = |f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)|$ satisfied the assumption $(H.6)$.

It follows from $(H.3)\,(ii)$ that $dP \times dt - a.e.$,

$$|\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)|^2 \leq c\,|\Delta Y_t^{n,m}|^2.$$

Then the assumption $(H.7)$ is satisfied for the generator $\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ of BDSDE (4.11) with $\lambda = c$, $\gamma = 0$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) with $\delta = 1$ that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq s\leq T} |\Delta Y_s^{n,m}|^2\right) + \mathbb{E}\left(\int_t^T |\Delta Z_s^{n,m}|^2 ds\right) \quad &\leq \quad 2\theta \int_t^T k\left(\mathbb{E} \sup_{s\leq u\leq T} |\Delta Y_u^{n,m}|^2\right) ds \qquad (4.12) \\ &\quad + \theta\mathbb{E}\int_t^T |f_n\left(s, Y_s^m, V_s\right) - f_m\left(s, Y_s^m, V_s\right)|^2 ds, \end{aligned}$$

where $\theta = K\exp\left(K\left(T - t\right)\right)$.

On one hand, it follows from (4.2) that, for each $n, m \geq 1$, $t \in [0, T]$ and each $\Delta Y_t^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.$,

$$\left|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)\right| \leq \int_{\{v:|v|\leq 1\}} \left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| \phi(v)\, dv,$$

and also from (4.5) and (4.9), we get

$$\begin{aligned} \left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| \quad &\leq \quad 2\left(\varphi\left(u+1\right) + c\beta + \beta\right) \\ &< \quad \infty. \end{aligned}$$

Using the continuity of f in y, we have

$$\lim_{n,m\to\infty} \left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| = 0,$$

applying Lebesgue's dominated convergence theorem, we get

$$\lim_{n,m\to\infty} \left|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)\right| = 0.$$

On the other hand, we obtain $dP \times dt - a.e.$,

$$\begin{aligned} \left|f_n(t, Y_t^m, V_t) - f_m(t, Y_t^m, V_t)\right| \quad &\leq \quad \int_{\{v:|v|\leq 1\}} 2\big(\varphi\big(u+1\big) + c\beta + \beta\big)\phi(v)\, dv, \\ &\leq \quad 2\big(\varphi\big(u+1\big) + c\beta + \beta\big) < \infty, \end{aligned}$$

applies again Lebesgue's dominated convergence theorem,yields that

$$\lim_{n,m\to\infty}\mathbb{E}\int_t^T \left|f_n\left(s,Y_s^m,V_s\right)-f_m\left(s,Y_s^m,V_s\right)\right|^2 ds=0. \tag{4.13}$$

Now, taking the lim sup in (4.12) and by Fatou's lemma, monotonicity and continuity of $k(\cdot)$ and (4.13) , we get

$$\begin{aligned}\lim_{n,m\to\infty}\sup\left(\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)\right) &\le 2\theta\int_t^T k(\lim_{n,m\to\infty}\sup\mathbb{E}\left(\sup_{s\le u\le T}|\Delta Y_u^{n,m}|^2\right)\\ &+ \mathbb{E}\left(\int_s^T|\Delta Z_u^{n,m}|^2\,du\right))ds.\end{aligned}$$

Thus, in view of $\int_{0^+} k^{-1}(u)\,du=\infty$, Bihari's inequality yields that, for each $0\le t\le T$

$$\lim_{n,m\to\infty}\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)=0,$$

which means that $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. Let $(Y_t,Z_t)_{t\in[0,T]}$ be the limit process of the sequence $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

On one hand, using (4.2) , (4.6) and (4.9) , we have

$$\begin{aligned}\left|f_n(s,Y_s^n,V_s)\right| &\le c\beta+\beta+\varphi\left(|Y^n|+1\right),\\ &\le c\beta+\beta+\varphi\left(u+1\right)<\infty,\end{aligned}$$

by definition of f_n and applying $(H.1)$, we have that f_n converge a.e. to f. Thus by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^T\left|f_n\left(s,Y_s^n,V_s\right)-f\left(s,Y_s,V_s\right)\right|ds=0.$$

In other hand, from the continuity properties of the stochastic integral, it follows that

$$\begin{cases}\sup_{0\le t\le T}\left|\int_t^T g\left(s,Y_s^n,V_s\right)d\overleftarrow{B}_s-\int_t^T g\left(s,Y_s,V_s\right)d\overleftarrow{B}_s\right|\to 0,\\[2ex] \sup_{0\le t\le T}\left|\int_t^T Z_s^n dW_s-\int_t^T Z_s dW_s\right|\to 0,\quad \text{as } n\to\infty \text{ in probability.}\end{cases}$$

from wich it follow that Y^n converge uniformly in t to Y i.e., $\lim_{n\to\infty}\left(\sup_{0\le t\le T}|Y_t^n-Y_t|\right)=0$. Finally, we pass to the limit $n\to\infty$ in (4.10) , we deduce that $(Y_t,Z_t)_{t\in[0,T]}$ solve BDSDE (4.4). ■

Lemma 4.2. *Let f and g satisfies the hypothesis **(H.1)–(H.5)**, $V\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ and $\xi\in\mathbb{L}^2\left(\mathcal{F}_T,\mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP-a.s.,\quad |\xi|\le\beta\quad dP\times dt-a.e.,\left|g\left(t,\omega,0,0\right)\right|\le\beta\quad and\quad \left|f\left(t,\omega,0,0\right)\right|\le\beta. \tag{4.14}$$

Then there exists a unique solution to the following BDSDE (4.4) .

Proof: In this lemma, we will eliminate the bounded condition with respecte to the processus $(V_t)_{t\in[0,T]}$ in Lemma 4.1. For each $n\ge 1$ and $Z\in\mathbb{R}^{k\times d}$, denote $q_n(Z)=\frac{Z\times n}{\sup(|Z|,n)}$, then $|q_n(Z)|=\left|\frac{Z\times n}{\sup(|Z|,n)}\right|\le\inf\left(|Z|,n\right)$. It follows from Lemma 4.1, that for each $n\ge 1$, there exists a solution $(Y_t^n,Z_t^n)_{t\in[0,T]}$ to the following BDSDE

$$Y_t^n=\xi+\int_t^T f\left(s,Y_s^n,q_n\left(V_s\right)\right)ds+\int_t^T g\left(s,Y_s^n,q_n\left(V_s\right)\right)d\overleftarrow{B}_s-\int_t^T Z_s^n dW_s,\quad 0\le t\le T. \tag{4.15}$$

In the sequel, we shall show that $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

In fact, for each $n\geq 1$ and $m\geq 1$, let $\Delta Y_t^{n,m}=Y_t^n-Y_t^m$ and $\Delta Z_t^{n,m}=Z_t^n-Z_t^m$. Then for each $0\leq t\leq T$

$$\Delta Y_t^{n,m}=\int_t^T \Delta f^{n,m}\left(s,\Delta Y_s^{n,m},V_s\right)ds+\int_t^T \Delta g^{n,m}\left(s,\Delta Y_s^{n,m},V_s\right)dB_s-\int_t^T \Delta Z_s^{n,m}dW_s, \quad (4.16)$$

where

$$\begin{cases}\Delta f^{n,m}\left(s,\Delta Y_s^{n,m},V_s\right)=f\left(s,\Delta Y_s^{n,m}+Y_s^m,q_n\left(V_s\right)\right)-f\left(s,Y_s^m,q_m\left(V_s\right)\right),\\ \Delta g^{n,m}\left(s,\Delta Y_s^{n,m},V_s\right)=g\left(s,\Delta Y_s^{n,m}+Y_s^m,q_n\left(V_s\right)\right)-g\left(s,Y_s^m,q_m\left(V_s\right)\right).\end{cases}$$

$(H.6)$ and $(H.7)$ is satisfied for the generator $\Delta f^{n,m}\left(t,\Delta Y_t^{n,m},V_t\right)$ with $\psi(u)=k(u)$, $\lambda=0$,

$$\sigma_t=\left|f\left(t,Y^m,q_n\left(V_t\right)\right)-f\left(t,Y^m,q_m\left(V_t\right)\right)\right|$$

respectively $\Delta g^{n,m}\left(t,\Delta Y_t^{n,m},V_t\right)$ with $\gamma=\alpha$ and $\eta_t=0$ of BDSDE (4.16).

Indeed by $(H.2)$, we get

$$\langle\Delta Y_t^{n,m},\Delta f\left({}^{n,m}t,\Delta Y_t^{n,m},V_t\right)\rangle\leq k\left(|\Delta Y_t^{n,m}|^2\right)+|\Delta Y_t^{n,m}|\left|f\left(t,Y^m,q_n\left(V_t\right)\right)-f\left(t,Y^m,q_m\left(V_t\right)\right)\right|.$$

and by $(H.3)\,(ii)$, we have

$$\left|\Delta g^{n,m}\left(t,\Delta Y_t^{n,m},V_t\right)\right|^2\leq c\left|\Delta Y_t^{n,m}\right|^2+\alpha\left|q_n(V_t)-q_m(V_t)\right|.$$

Thus, it follow from Proposition 3.1 (i) with $\delta=1$ that there exists a constant $K>0$ depending only on δ,λ and γ such that, for each $0\leq t\leq T$

$$\mathbb{E}(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2)+\mathbb{E}(\int_t^T|\Delta Z_s^{n,m}|^2\,ds)\leq \qquad (2K\int_t^T k(\mathbb{E}\,|\Delta Y_s^{n,m}|^2)ds$$
$$+\quad K\mathbb{E}\int_t^T\left|f(s,Y_s^m,q_n(V_s))-f(s,Y_s^m,q_m(V_s))\right|^2ds)\exp(K\,(T-t))$$

using $(H.3)\,(i)$ and $\theta=K\exp\left(K\left(T-t\right)\right)$, we get

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)\leq 2\theta\int_t^T k\left(\mathbb{E}\,|\Delta Y_s^{n,m}|^2\right)ds+\theta c\mathbb{E}\int_t^T\left|q_n\left(V_s\right)-q_m\left(V_s\right)\right|^2ds. \quad (4.17)$$

since $k\left(x\right)\leq A\left(1+x\right)$, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq 2\theta AT+2\theta A\int_t^T\mathbb{E}\left(\sup_{s\leq u\leq T}|\Delta Y_u^{n,m}|^2\right)ds\\ &+\theta c\mathbb{E}\int_t^T\left|q_n\left(V_s\right)-q_m\left(V_s\right)\right|^2ds.\end{aligned}$$

Applying Gronwall's Lemma and $(a-b)^2\leq a^2+b^2$, yields that for each $t\in[0,T]$ and each $n,m\geq 1$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \left(2\theta AT+\theta c\mathbb{E}\int_t^T\left(|q_n\left(V_s\right)|^2+|q_m\left(V_s\right)|^2\right)ds\right)\exp\left(2\theta AT\right),\\ &\leq \left(2\theta AT+2\theta c\mathbb{E}\int_0^T|V_s|^2\,ds\right)\exp\left(2\theta AT\right).\end{aligned}$$

By taking the lim sup in (4.17), we have

$$\begin{aligned}\lim\sup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \lim\sup_{n,m\to\infty}(2\theta\int_t^T k\left(\mathbb{E}\,|\Delta Y_s^{n,m}|^2\right)ds\\ &+\theta c\mathbb{E}\int_t^T|q_n(V_s)-q_m(V_s)|^2\,ds),\end{aligned}$$

by continuity and monotonicity of $k(\cdot)$, Fatou's lemma, we have

$$\begin{aligned}\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2ds\right) &\le 2\theta\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\,|\Delta Y_s^{n,m}|^2\right)ds\\ &+ \theta c\mathbb{E}\int_t^T\limsup_{n,m\to\infty}|q_n(V_s)-q_m(V_s)|^2ds.\end{aligned}$$

since

$$\mathbb{E}\int_t^T\limsup_{n,m\to\infty}|q_n(V_s)-q_m(V_s)|^2ds=0.$$

Thus, in view of $\int_{0^+}k^{-1}(u)\,du=\infty$, Bihari's inequality yields that, for each $0\le t\le T$

$$\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2ds\right)=0.$$

We know that $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. Let $(Y_t,Z_t)_{t\in[0,T]}$ be the limit of the sequence $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

Applying $(H.1)$, $(H.3)\,(i)$, $(H.4)$, (4.14) and Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^T\left|f(s,Y_s^n,q_n(V_s))-f(s,Y_s,V_s)\right|ds=0,$$

from wich it follow that Y^n converge uniformly in t to Y i.e, $\lim_{n\to\infty}\left(\sup_{0\le t\le T}|Y_t^n-Y_t|\right)=0$. Finally, we pass to the limit $n\to\infty$ in (4.15), we deduce that $(Y_t,Z_t)_{t\in[0,T]}$ solve BDSDE (4.4). ■

Lemma 4.3. *Let f and g satisfies the hypothesis **(H.1)–(H.5)** and $\xi\in\mathbb{L}^2\left(\mathcal{F}_T,\mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP-a.s.,\left|\xi\right|\le\beta\quad dP\times dt-a.e.,\left|g(t,\omega,0,0)\right|\le\beta\quad\text{and}\quad\left|f(t,\omega,0,0)\right|\le\beta.\qquad(4.18)$$

Then there exists a unique solution to the following BDSDE $\left(E^{\xi,f,g}\right)$.

Proof: By Lemma 4.2, we can construct the iterative sequence. Let us set as usual $\left(Y_t^0,Z_t^0\right)=(0,0)$ and define recursively, for each $n\ge1$

$$Y_t^n=\xi+\int_t^T f\left(s,Y_s^n,Z_s^{n-1}\right)ds+\int_t^T g\left(s,Y_s^n,Z_s^{n-1}\right)d\overleftarrow{B}_s-\int_t^T Z_s^n dW_s,\qquad t\in[0,T].\quad(4.19)$$

It follows from $(H.2)$ and $(H.3)\,(i)$ that $dP\times dt-a.e.$,

$$\begin{aligned}\langle Y_s^n,f\left(s,Y_s^n,Z_s^{n-1}\right)\rangle &= \langle Y_s^n,f\left(s,Y_s^n,Z_s^{n-1}\right)-f\left(s,0,Z_s^{n-1}\right)+f\left(s,0,Z_s^{n-1}\right)\rangle,\\ &\le k\left(|Y_s^n|^2\right)+|Y_s^n|\left(c\left|Z_s^{n-1}\right|+|f(s,0,0)|\right),\end{aligned}$$

then the assumption $(H.6)$ is satisfied for the generator $f\left(s,Y_s^n,Z_s^{n-1}\right)$ of BDSDE (4.19) with $\psi(u)=k(u)$, $\lambda=0$, $\sigma_t=c\left|Z_t^{n-1}\right|+|f(t,0,0)|$.

It follows from $(H.3)\,(ii)$ that $dP\times dt-a.e.$,

$$\left|g\left(t,Y_t^n,Z_t^{n-1}\right)\right|^2\le 2c\,|Y_t^n|^2+2\alpha^2\left|Z_t^{n-1}\right|^2+2\,|g(t,0,0)|^2,$$

then the generator $g\left(t, Y_t^n, Z_t^{n-1}\right)$ of BDSDE (4.19) with $\gamma = 2\alpha^2$, $\lambda = 2c$ and $\eta_t = 2\left|g(t,0,0)\right|^2$. satisfied the assumption $(H.7)$.

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right) &\leq \left(K\mathbb{E}\left|\xi\right|^2+2K\int_t^T k\left(\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)\right)ds\right.\\ &+ \frac{K}{\delta}\mathbb{E}\int_t^T\left(c\left|Z_s^{n-1}\right|+\left|f(s,0,0)\right|\right)^2ds\\ &\left.+2K\mathbb{E}\int_t^T\left|g(s,0,0)\right|^2ds\right)\exp\left(K(T-t)\right).\end{aligned}$$

By $\theta = K\exp\left(K\left(T-t\right)\right)$, we note $H(t) = \theta\left(\mathbb{E}\left|\xi\right|^2+\frac{2}{\delta}\mathbb{E}\int_t^T\left|f\left(s,0,0\right)\right|^2ds+2\mathbb{E}\int_t^T\left|g\left(s,0,0\right)\right|^2ds\right)$. Using (4.18), we have $H(t) \leq \theta\beta^2\left(1+\frac{2T}{\delta}+2T\right) = \theta h$. Therefore

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right)\leq\theta h+2\theta\int_t^T k\left(\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)\right)ds+\frac{2\theta c^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Since $k\left(x\right) \leq A\left(1+x\right)$, we get

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right)\quad\leq\quad\theta h+2A\theta T+2A\theta\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{2\theta c^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Let us set $\vartheta_1 = \max\left\{T-\frac{\ln 2}{K}, T-\frac{\ln 2}{4KA}, 0\right\}$. Then for each $t \in [\vartheta_1, T]$,we have $\exp\left(K(T-t)\right) \leq 2$, thus $\theta \leq 2K$ and

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right)\quad\leq\quad 2Kh+4KAT+4AK\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{4Kc^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds,$$

we take $\delta = 16Kc^2$, obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right)\leq 2Kh+4KAT+4AK\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{1}{4}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Applying Growall's lemma yields that for each $t \in [\vartheta_1, T]$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right)\leq\left(2Kh+4KAT+\frac{1}{4}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds\right)\exp\left(4AK\left(T-t\right)\right).$$

For each $t \in [\vartheta_1, T]$, we have $\exp\left(4AK\left(T-t\right)\right) < 2$, then we deduce for each $n \geq 1$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right) &\leq 4Kh+8KAT+\frac{1}{2}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds,\\ &\leq 8Kh+16KAT. \qquad (4.20)\end{aligned}$$

In the sequel, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$ and $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$. Then $\forall t \in [0, T]$

$$\Delta Y_t^{n,m}=\int_t^T\Delta f^{n,m}\left(s,\Delta Y_s^{n,m}\right)ds+\int_t^T\Delta g^{n,m}\left(s,\Delta Y_s^{n,m}\right)dB_s-\int_t^T\Delta Z_s^{n,m}dW_s, \qquad (4.21)$$

where

$$\begin{cases}\Delta f^{n,m}\left(s,\Delta Y_s^{n,m}\right)=f\left(s,\Delta Y_s^{n,m}+Y_s^m,Z_s^{n-1}\right)-f\left(s,Y_s^m,Z_s^{m-1}\right),\\ \Delta g^{n,m}\left(s,\Delta Y_s^{n,m}\right)=g\left(s,\Delta Y_s^{n,m}+Y_s^m,Z_s^{n-1}\right)-g\left(s,Y_s^m,Z_s^{m-1}\right).\end{cases}$$

It follows from $(H.2)$ and $(H.3)$ that $dP \times dt - a.e.$,

$$\begin{aligned} \langle \Delta Y_t^{n,m}, \Delta f^{n,m}\left(t, \Delta Y_t^{n,m}\right)\rangle &= \langle \Delta Y_t^{n,m}, f\left(s, \Delta Y_t^{n,m} + Y_t^m, Z^{n-1}\right) - f\left(s, Y_s^m, Z^{m-1}\right)\rangle, \\ &\leq k\left(\left|\Delta Y_t^{n,m}\right|^2\right) + \left|\Delta Y_t^{n,m}\right| \left|f\left(t, Y_t^m, Z_t^{n-1}\right) - f\left(t, Y_t^m, Z_t^{m-1}\right)\right|. \end{aligned}$$

Then the the generator $\Delta f^{n,m}\left(t, \Delta Y_t^{n,m}\right)$ of BDSDE (4.21) with

$$\psi(u) = k(u), \lambda = 0, \sigma_t = \left|f\left(t, Y_t^m, Z_t^{n-1}\right) - f\left(t, Y_t^m, Z_t^{m-1}\right)\right|.$$

satisfied the assumption $(H.6)$.

It follows from $(H.3)(ii)$ that $dP \times dt - a.e.$,

$$\left|\Delta g^{n,m}\left(s, \Delta Y_s^{n,m}\right)\right|^2 \leq c\left|\Delta Y_s^{n,m}\right|^2 + \alpha\left|\Delta Z_s^{n-1,m-1}\right|.$$

Then the assumption $(H.7)$ is satisfied for the generator $\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}\right)$ of BDSDE (4.21) with, $c = \lambda$, $\alpha = \gamma$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right) &\leq \exp(K(T-t))\Big(2K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right) ds \\ &+ \frac{K}{\delta}\mathbb{E}\int_t^T \left|f(s, Y_s^m, Z_s^{n-1}) - f(s, Y_s^m, Z_s^{m-1})\right|^2 ds\Big). \end{aligned}$$

Let us set $\vartheta_1 = \max\left\{T - \frac{\ln 2}{K}, 0\right\}$. Then for each $t \in [\vartheta_1, T]$, we have $\exp\left(K(T-t)\right) \leq 2$ and

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right) \leq 4K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right) ds + \frac{2Kc^2}{\delta}\mathbb{E}\int_t^T \left|\Delta Z_s^{n-1,m-1}\right|^2 ds,$$

take $\delta = 8Kc^2$, we have

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right) &\leq 4K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right) ds \\ &+ \frac{1}{4}\mathbb{E}\int_t^T \left|\Delta_s Z^{n-1,m-1}\right|^2 ds. \qquad (4.22) \end{aligned}$$

Using monotonicity and continuity of $k(\cdot)$, (4.20), and taking the $\limsup$ in (4.22), by Fatou's lemma, we have

$$\begin{aligned} \limsup_{n,m\to\infty}\left(\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right)\right) &\leq 4K\int_t^T k\left(\lim\sup_{n,m\to\infty}\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right) ds \\ &+ \frac{1}{4}\mathbb{E}\int_t^T \lim\sup_{n,m\to\infty}\left|\Delta Z_s^{n-1,m-1}\right|^2 ds. \end{aligned}$$

Thus, in view of $\int_{0^+} k^{-1}(u)\,du = \infty$, Bihari's inequality yields that, for each $\vartheta_1 \leq t \leq T$

$$\lim\sup_{n,m\to\infty}\left(\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right)\right) = 0,$$

we know that $\left((Y_t^n, Z_t^n)_{t\in[\vartheta_1,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(\vartheta_1, T, \mathbb{R}^k\right) \times \mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$. Let $(Y_t, Z_t)_{t\in[\vartheta_1,T]}$ be the limit of the sequence $\left((Y_t^n, Z_t^n)_{t\in[\vartheta_1,T]}\right)_{n\in\mathbb{N}^*}$ in the space $\mathcal{S}^2\left(\vartheta_1, T, \mathbb{R}^k\right) \times \mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$. On the other hand, since Z_t^n converge in $\mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$ to Z_t, then there exists a subsequence which will denote Z_t^n such that $\forall n$, $Z_t^n \to Z_t$, $dt \otimes dP - a.s.$ and $\sup_n |Z_t^n|$ is $dt \otimes dP$ integrable. Therefore by $(H.3)\,(i)$ and $(H.4)$, we have

$$\left|f\left(s, Y_s^n, Z_s^{n-1}\right)\right| \leq c\left|Z_s^{n-1}\right| + \left|f(s,0,0)\right| + \varphi\left(\left|Y_s^n\right|\right) < \infty,$$

applying $(H.1)$ and $(H.3)\,(i)$, we have

$$\begin{aligned} \lim_{n\to\infty}\left|f\left(s, Y_s^n, Z_s^{n-1}\right) - f(s, Y_s, Z_s)\right| &= \lim_{n\to\infty}\left|f\left(s, Y_s, Z_s^{n-1}\right) - f(s, Y_s, Z)\right| \\ &\leq c\lim_{n\to\infty}\left|Z_s^{n-1} - Z\right| = 0, \end{aligned}$$

thus, $f(s, Y_s^n, Z_s^{n-1})$ converge a.e. to $f(s, Y_s, Z_s)$. Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_t^T \left|f\left(s, Y_s^n, Z_s^{n-1}\right) - f(s, Y_s, Z_s)\right| ds = 0.$$

From wich it follow that Y^n converge uniformly in $t \in [\vartheta_1, T]$ to Y i.e, $\lim_{n\to\infty}\left(\sup_{\vartheta_1 \leq t \leq T}|Y_t^n - Y_t|\right) = 0$. Now, we pass to the limit $n \to \infty$ in (4.19), we follows that $(Y_t, Z_t)_{t\in[\vartheta_1,T]}$ solve BDSDE $\left(E^{\xi,f,g}\right)$.

Note that $T - \vartheta_1 \geq 0$ and depends only on c and A, we can repeat the above operation in finite steps to obtain a solution to the BDSDE $\left(E^{\xi,f,g}\right)$ on $[\vartheta_2, \vartheta_1]$, $[\vartheta_3, \vartheta_2]$, ..., and then on $[0, T]$. ■

Now, proof of Theorem 4.1. Firstly we approximate $f(t, Y_t, Z_t)$ and ξ by a sequence whose elements satisfy the bound assumption in Lemma 4.3.

For each $n \geq 1$, define $q_n(x) = \frac{x \times n}{\sup(|x|, n)}$ for each $x \in \mathbb{R}^k$, and let

$$\xi_n = q_n(\xi) \quad \text{and} \quad f_n(t, Y_t, Z_t) = f(t, Y_t, Z_t) - f(t,0,0) + q_n(f(t,0,0)), \tag{4.23}$$

clearly, the f_n satisfies (4.18), we have

$$\lim_{n\to\infty}\mathbb{E}\left|\xi_n - \xi\right|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty}\mathbb{E}\left(\int_0^T \left|q_n(f(s,0,0)) - f(s,0,0)\right|^2 ds\right) = 0. \tag{4.24}$$

For each $n \geq 1$, let $(Y_t^n, Z_t^n)_{t\in[0,T]}$ denote the unique solution to the following BDSDE

$$Y_t^n = \xi_n + \int_t^T f_n(s, Y_s^n, Z_s^n)\,ds + \int_t^T g(s, Y_s^n, Z_s^n)\,d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \qquad 0 \leq t \leq T. \tag{4.25}$$

In the sequel, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$ and $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$. Then $\forall t \in [0, T]$

$$\begin{aligned} \Delta Y_t^{n,m} &= \xi_n - \xi_m + \int_t^T \Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\,ds \\ &\quad + \int_t^T \Delta g^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\,dB_s - \int_t^T \Delta Z_s^{n,m} dW_s, \end{aligned} \tag{4.26}$$

where

$$\begin{cases} \Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}) = f_n(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - f_m(s, Y_s^m, Z_s^m), \\ \Delta g^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}) = g(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - g(s, Y_s^m, Z_s^m). \end{cases}$$

By add and subtract, we get

$$\begin{aligned} \langle \Delta Y_s^{n,m}, \Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\rangle &= \langle \Delta Y_s^{n,m}, f_m(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - f_m(s, Y_s^m, Z_s^m)\rangle \\ &\quad + \langle \Delta Y_s^{n,m}, f_n(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) \\ &\quad - f_m(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m)\rangle. \end{aligned}$$

It follows from $(H.2)$ and $(H.3)\,(i)$ and (4.23) that $dP\times dt-a.e.$,

$$\begin{aligned}
&\langle \Delta Y_s^{n,m}, \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}\right)\rangle \\
=\ & \langle \Delta Y_s^{n,m}, f\left(s, \Delta Y_s^{n,m}+Y_s^m, \Delta Z_s^{n,m}+Z_s^m\right)-f\left(s, Y_s^m, Z_s^n\right)-f\left(s, Y_s^m, Z_s^m\right)+f\left(s, Y_s^m, Z_s^n\right)\rangle \\
&+\langle \Delta Y_s^{n,m}, q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)\rangle, \\
\leq\ & k\left(|\Delta Y_s^{n,m}|^2\right)+c\,|\Delta Z_s^{n,m}|\,|\Delta Y_s^{n,m}|+|\Delta Y_s^{n,m}|\,|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|\,.
\end{aligned}$$

Then the assumption $(H.6)$ is satisfied for the generator $\Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}\right)$ of BDSDE (4.26) with $\psi\left(u\right)=k\left(u\right),\ \lambda=c,\ \sigma_t=|q_n\left(f\left(t,0,0\right)\right)-q_m\left(f\left(t,0,0\right)\right)|\,.$

It follows from $(H.3)\,(ii)$ that $dP\times dt-a.e.$,

$$\begin{aligned}
\left|\Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}\right)\right|^2 &= \left|g\left(s, \Delta Y_s^{n,m}+Y_s^m, \Delta Z_s^{n,m}+Z_s^m\right)-g\left(s, Y_s^m, Z_s^m\right)\right|^2, \\
&\leq c\left|\Delta Y_s^{n,m}\right|^2+\alpha\left|\Delta Z_s^{n,m}\right|^2.
\end{aligned}$$

Then the generator $\Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}\right)$ of BDSDE (4.26) with, $\lambda=c,\ \alpha=\gamma$ and $\eta_t=0.$ satisfied the assumption $(H.7)\,.$

Thus, it follow from Proposition 3.1 (i) with $\delta=1$ that there exists a constant $K>0$ depending only on δ,λ and γ such that, for each $0\leq t\leq T$

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \theta\mathbb{E}\,|\xi_n-\xi_m|^2+2\theta\int_t^T k\left(\mathbb{E}\left(\sup_{0\leq r\leq s}|\Delta Y_r^{n,m}|^2\right)\right)ds \\
&\quad +\theta\mathbb{E}\int_t^T|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|^2\,ds. \quad (4.27)
\end{aligned}$$

where $\theta=K\exp\left(K\left(T-t\right)\right).$ Since $k\left(x\right)\leq A\left(1+x\right)$, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \theta\mathbb{E}\,|\xi_n-\xi_m|^2+2AT\theta+2A\theta\int_t^T\mathbb{E}\left(\sup_{0\leq r\leq s}|\Delta Y_r^{n,m}|^2\right)ds \\
&\quad +\theta\mathbb{E}\int_t^T|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|^2\,ds.
\end{aligned}$$

Using (4.24), we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq 2\theta\mathbb{E}\,|\xi|^2+2AT\theta+2A\theta\int_t^T\mathbb{E}\left(\sup_{0\leq r\leq s}|\Delta Y_r^{n,m}|^2\right)ds \\
&\quad + 2\theta\mathbb{E}\int_t^T|f\left(s,0,0\right)|^2\,ds.
\end{aligned}$$

Applying Growall's lemma yields that for each $t\in[0,T]$ and each $n,m\geq1$

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \left(2\theta AT+2\theta\mathbb{E}\,|\xi|^2+2\theta\mathbb{E}\int_t^T|f\left(s,0,0\right)|^2\,ds\right)\exp\left(2\theta AT\right), \\
&< \infty.
\end{aligned}$$

Taking the $\limsup$ in (4.27) and by previous inequality, Fatou's lemma, monotonicity and continuity of $k\left(\cdot\right)$, we have

$$\begin{aligned}
\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{t\leq r\leq T}|\Delta Y_r^{n,m}|^2+\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) &\leq \theta\mathbb{E}\left(\limsup_{n,m\to\infty}|\xi_n-\xi_m|^2\right) \\
&\quad + 2\theta\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{s\leq r\leq T}|\Delta Y_r^{n,m}|^2\right)\right)ds \\
&\quad +\theta\mathbb{E}\int_t^T\limsup_{n,m\to\infty}|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|^2\,ds, \\
&= 2\theta\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{s\leq r\leq T}|\Delta Y_r^{n,m}|^2\right)\right)ds.
\end{aligned}$$

Thus, in view of $\int_{0+} k^{-1}(u)\, du = \infty$, Bihari's inequality yields that for each $0 \leq t \leq T$

$$\limsup_{n,m\to\infty} \mathbb{E}\left(\sup_{t\leq r\leq T} |\Delta Y_r^{n,m}|^2 + \int_t^T |\Delta Z_s^{n,m}|^2\, ds\right) = 0.$$

We know that $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. Let $(Y_t, Z_t)_{t\in[0,T]}$ be the limit of the sequence $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. Using $(H.3)\,(i)$ and $(H.4)$, we have

$$\begin{aligned} \left|f_n\left(t, Y_t^n, Z_t^n\right)\right| &= c\left|Z_t^n\right| + \left|f\left(t,0,0\right)\right| + \varphi\left(\left|Y_t^n\right|\right), \\ &< \infty, \end{aligned}$$

applying $(H.1)$, $(H.3)$ and (4.23), we have $f_n(s, Y_s^n, Z_s^n)$ converge a.e. to $f(s, Y_s, Z_s)$. Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_t^T \left|f_n\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s, Z_s\right)\right| ds = 0.$$

from wich it follow that Y^n converge uniformly in t to Y. Now, we pass to the limit $n\to\infty$ in (4.25), we deduce that $(Y_t, Z_t)_{t\in[0,T]}$ solve BDSDE $(E^{\xi,f,g})$.

Thus we complete the proof of Theorem 4.1. ■

5 Application to SPDEs

In this section we connect BDSDEs with weak monotonicity and general growth generators with the correspondent SPDEs and give the Sobolev solution of the SPDEs.

Notation and Definition: C_b^k set of function of class C^k, whose partial derivatives of order less then or equal to k are bounded. Given $x \in \mathbb{R}^d$, $b \in C_b^2\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and $\sigma \in C_b^3\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$, denote by $\left(X_s^{t,x};\ t \leq s \leq T\right)$ the unique strong solution of the SDEs following

$$dX_s^{t,x} = b\left(X_s^{t,x}\right) ds + \sigma\left(X_s^{t,x}\right) dW_s, \qquad X_t^{t,x} = x. \tag{5.1}$$

It's well know that $\mathbb{E}\left(\sup_{t\leq s\leq T}\left|X_s^{t,x}\right|^p\right) < \infty$ for any $p > 1$, we recall that the stochastic flow associated to the diffusion processus $\left(X_s^{t,x};\ t \leq s \leq T\right)$ is $\left(X_s^{t,x};\ x \in \mathbb{R}^d,\ t \leq s \leq T\right)$ and the inverse flow is denote by $\hat{X}_s^{t,x}$. $x \to \hat{X}_s^{t,x}$ is differentiable and we denote by $J\left(\hat{X}_s^{t,x}\right)$ the determinant of the Jacobian matrix of $\hat{X}_s^{t,x}$, which is positive and satisfies $J\left(\hat{X}_s^{t,x}\right) = 1$.

For $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$ we define the process $\phi_t : \Omega\times[0,T]\times\mathbb{R}^d \to \mathbb{R}$ by $\phi_t\left(s,x\right) = \phi\left(\hat{X}_s^{t,x}\right) J\left(\hat{X}_s^{t,x}\right)$.

Let $\pi : \mathbb{R}^d \to \mathbb{R}_+$ be an integrable continues positif function and $\mathbb{L}^2\left(\mathbb{R}^d, \pi\left(x\right) dx\right)$ be the weight $\mathbb{L}^2$ space with weight $\pi\left(x\right)$ endowed with the following norm

$$||u||_\pi^2 = \int_{\mathbb{R}^d} |u\left(x\right)|^2 \pi\left(x\right) dx.$$

Let us take the weight $\pi\left(x\right) = \exp\left(F\left(x\right)\right)$, where $F : \mathbb{R}^d \to \mathbb{R}$ is a continues, moreover we assume that there exist some $R > 0$ such that $F \in C_b^2$ for $|x| > R$, we need the following result of generalized equivalence of norm.

Lemma 5.1. *There exist two positive constants K_1, k_1 which depend on T, π, such that for any $t \leq s \leq T$ and $\Phi \in L^1\left(\Omega \times \mathbb{R}^d, \mathbb{P}\otimes\pi(x)\,dx\right)$*

$$k_1\left(\int_{\mathbb{R}^d}|\Phi(x)|\,\pi(x)\,dx\right) \leq \mathbb{E}\left(\int_{\mathbb{R}^d}\left|\Phi\left(X_s^{t,x}\right)\right|\pi(x)\,dx\right) \leq K_1\left(\int_{\mathbb{R}^d}|\Phi(x)|\,\pi(x)\,dx\right).$$

Moreover for any $\Psi \in L^1\left(\Omega \times [0,\ T] \times \mathbb{R}^d \times \ \mathbb{P}\otimes dt \otimes \pi(x)\,dx\right)$

$$\begin{aligned} k_1\left(\int_{\mathbb{R}^d}\int_t^T|\Psi(s,x)|\,ds\ \pi(x)\,dx\right) &\leq \mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left|\Psi\left(s,X_s^{t,x}\right)\right|ds\ \pi(x)\,dx\right), \\ &\leq K_1\left(\int_{\mathbb{R}^d}\int_t^T|\Psi(s,x)|\,ds\ \pi(x)\,dx\right). \end{aligned}$$

Proof: Using the change of variable $y = X_s^{t,x}$, we get

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{R}^d}\left|\Phi\left(X_s^{t,x}\right)\right|\pi(x)\,dx\right) &= \int_{\mathbb{R}^d}|\Phi(y)|\,\mathbb{E}\left(\pi\left(\hat{X}_s^{t,y}\right)J\left(\hat{X}_s^{t,y}\right)\right)dy, \\ &= \int_{\mathbb{R}^d}\mathbb{E}\left(\frac{|\Phi(y)|\,\pi\left(\hat{X}_s^{t,y}\right)}{\pi(y)}\right)\pi(y)\,dy. \end{aligned}$$

By Lemma 5.1 in Bally-Matoussi [5], $k_1 \leq \mathbb{E}\left(\frac{|\Phi(y)|\pi\left(\hat{X}_s^{t,y}\right)}{\pi(y)}\right) \leq K_1$ for any $y \in \mathbb{R}^k$, $s \in [t,T]$, the first claim follows. The second claim can be proved similarly. ■

Now begin to study the following SPDEs

$$(\mathcal{P}^{(f,g)}) \qquad \begin{cases} u(t,x) = h(x) + \displaystyle\int_s^T \left(\mathcal{L}u(r,x) + f\left(r,x,u(r,x),\sigma^*\nabla u(r,x)\right)\right)dr \\ \qquad + \displaystyle\int_s^T g\left(r,x,u(r,x),\sigma^*\nabla u(r,x)\right)d\overleftarrow{B}_r, \quad t \leq s \leq T, \end{cases}$$

where

$$\mathcal{L} := \frac{1}{2}\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i\partial x_j} + \sum_i b_i\frac{\partial}{\partial x_i}, \qquad \text{with } (a_{ij}) := \sigma\sigma^*.$$

Let $\mathcal{H}$ be the set of random fields $\left\{u(t,x), 0 \leq t \leq T,\ x \in \mathbb{R}^d\right\}$ such that for every (t,x), $u(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable and

$$\|u\|_{\mathcal{H}}^2 = E\Big(\int_{\mathbb{R}^d}\int_0^T(|u(r,x)|^2 + |(\sigma^*\nabla u)(r,x)|^2)dr\ \pi(x)\,dx\Big) < \infty.$$

The couple $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Banach space.

Definition 5.1. *We say that u is a Sobolev solution to SPDE $(\mathcal{P}^{(f,g)})$, if $u \in \mathcal{H}$ and for any $\varphi \in \mathcal{C}_c^{1,\infty}\left([0,T] \times \mathbb{R}^d\right)$*

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_s^T u(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)\,drdx + \int_{\mathbb{R}^d}u(r,x)\,\varphi(r,x)\,dx - \int_{\mathbb{R}^d}h(x)\,\varphi(T,x)\,dx \\ &-\frac{1}{2}\int_{\mathbb{R}^d}\int_s^T\sigma^*u(r,x)\,\sigma^*\varphi(r,x)\,drdx - \int_{\mathbb{R}^d}\int_s^T u\,div\big(\left(b-A\right)\varphi\big)(r,x)\,drdx \\ &= \int_{\mathbb{R}^d}\int_s^T f\left(r,x,u(r,x),\sigma^*\nabla u(r,x)\right)\varphi(r,x)\,drdx + \int_{\mathbb{R}^d}\int_s^T g\left(r,x,u(r,x),\sigma^*\nabla u(r,x)\right)\varphi(r,x)\,d\overleftarrow{B}_r dx \end{aligned} \tag{5.2}$$

where A is a d-vector whose coordinates are defined by $A_j := \frac{1}{2}\sum_{i=1}^d\frac{\partial a_{ij}}{\partial x_i}$.

In this section well study the Sobolev solution of $\left(\mathcal{P}^{(f,g)}\right)$ with weak monotonicity and general growth. For $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$, $g:[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^{k\times l}$, $h:\mathbb{R}^d\to\mathbb{R}^d$.

The main idea is to connect $\left(\mathcal{P}^{(f,g)}\right)$ with the following BDSDE for each $s\in[t,T]$

$$\begin{aligned} Y_s^{t,x} &= h\left(X_T^{t,x}\right)+\int_s^T f\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)dr \\ &\quad +\int_s^T g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)d\overleftarrow{B}_r-\int_s^T Z_r^{t,x}dW_r, \end{aligned} \tag{5.3}$$

where $\left(X_s^{t,x};0\le s\le T\right)$ is the solution of SDEs (5.1).
Our object consists to establish the existence and uniqueness of solutions u to SPDEs $\left(\mathcal{P}^{(f,g)}\right)$ such that $u\left(s,X_s^{t,x}\right)=Y_s^{t,x}$ and $\sigma^*\nabla u\left(s,X_s^{t,x}\right)=Z_s^{t,x}$.

We consider the following assumptions **(A)**:
(A.1) For (t,x) fixed $dP\times dt$-a.e., $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{k\times d}$ $y\to f(w,t,x,y,z)$ is continuous and

$$\int_{\mathbb{R}^d}\int_0^T|f(t,x,0,0)|^2\,dt\pi(x)\,dx<\infty.$$

(A.2) f satisfies the weak monotonicity condition in y, i.e., there exist a nondecreasing and concave function $k(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ with $k(u)>0$ for $u>0$, $k(0)=0$ and $\int_{0^+}k(u)du=+\infty$ such that $dP\times dt$-a.e.,$\forall y_1,y_2\in\mathbb{R}^k$, $z\in\mathbb{R}^{k\times d}$, $x\in\mathbb{R}^d$

$$\langle y_1-y_2,f(t,\omega,x,y_1,z)-f(t,\omega,x,y_2,z)\rangle\le k\left(|y_1-y_2|^2\right).$$

(A.3) i) f is Lipschitz in z, uniformly with respect to (ω,t,x,y) i.e., there exists a constant $c>0$ such that $dP\times dt$-a.e.,

$$\left|f(\omega,t,x,y,z)-f\left(\omega,t,x,y,z'\right)\right|\le c|z-z'|.$$

ii) $\int_{\mathbb{R}^d}\int_0^T|g(t,x,0,0)|^2\,dt\pi(x)\,dx<\infty$ and for (t,x) fixed there exists a constant $c>0$ and a constant $0<\alpha\le\frac{1}{4}$ such that $dP\times dt$-a.e.,

$$\left|g(\omega,t,x,y,z)-g\left(\omega,t,x,y',z'\right)\right|\le c|y-y'|+\alpha|z-z'|.$$

(A.4) f have a general growth with respect to y, i.e., $dP\times dt$-a.e., $\forall(x,y)\in\mathbb{R}^d\times\mathbb{R}^k$

$$\left|f(t,\omega,x,y,0)\right|\le\left|f(t,\omega,x,0,0)\right|+\varphi(|y|),$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is increasing continuous function.
(A.5) h belongs to $\mathbb{L}^2\left(\mathbb{R}^d,\pi(x)\,dx;\mathbb{R}^d\right)$.

Now by Lemma 5.1, Fubini's theorem and using $(A.1)$, $(A.3)(ii)$ and $(A.5)$, we have for a.e. $x\in\mathbb{R}^d$

$$\mathbb{E}\left(\int_s^T\left|f\left(r,X_r^{t,x},0,0\right)\right|^2dr+\int_s^T\left|g\left(r,X_r^{t,x},0,0\right)\right|^2dr+\left|h\left(X_T^{t,x}\right)\right|^2\right)<\infty. \tag{5.4}$$

Hence, it follows from Theorem 4.1, that BDSDEs (5.3) admit a unique solution $\left(Y_s^{t,x},Z_s^{t,x}\right)$ such that $Y_s^{t,x},Z_s^{t,x}$ are $\mathcal{F}_{t,s}^W\vee\mathcal{F}_{s,T}^B$ measurable for any $s\in[0,T]$.

Moreover, by Proposition 3.1 (i) it's easy to check for each $\delta > 0$ that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|Z_s^{t,x}\right|^2 ds\right) &\leq \left(\mathbb{E}|h\left(X_T^{t,x}\right)|^2+2\int_t^T k\left(\mathbb{E}|Y_s^{t,x}|^2\right)ds+\frac{1}{\delta}\mathbb{E}\int_t^T\left|f\left(s,X_s^{t,x},0,0\right)\right|^2 ds\right.\\ &\quad\left.+2\mathbb{E}\int_t^T g\left(s,X_s^{t,x},0,0\right)ds\right)K\exp\left(K\left(T-T\right)\right),\end{aligned}$$

using (5.4) and since $k(x) \leq A(1+x)$, we have

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2+\int_t^T\left|Z_s^{t,x}\right|^2 ds\right)\leq c+2\theta AT+2\theta A\int_t^T\mathbb{E}\left(|Y_s^{t,x}|^2\right)ds,$$

where $\theta = K\exp(K(T-T))$. Finally, applying Gronwall's lemma, we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2+\int_t^T\left|Z_s^{t,x}\right|^2 ds\right)\leq (c+2\theta AT)\exp(2\theta AT)<\infty. \tag{5.5}$$

Now, we are state the main result of this section.

Theorem 5.1. *Under hypothesis (A), the SPDEs* $\left(\mathcal{P}^{(f,g)}\right)$ *admits a unique Sobolev solution u. Moreover* $u(t,x) = Y_t^{t,x}$*, where* $\left(Y_s^{t,x}, Z_s^{t,x}\right)_{t\leq s\leq T}$ *is the unique solution of the BDSDEs (5.3) and*

$$u\left(s,X_s^{t,x}\right)=Y_s^{t,x}\quad\text{and}\quad\left(\sigma^*\nabla u\right)\left(s,X_s^{t,x}\right)=Z_s^{t,x},\quad\text{for a.e. }(s,\omega,x)\text{ in }[t,T]\times\Omega\times\mathbb{R}^d. \tag{5.6}$$

We first consider the following SPDEs:

$$\left(\mathcal{P}^{(f,g,u^n)}\right)\qquad\begin{cases}u^n(t,x)=h(x)+\displaystyle\int_s^T\left(\mathcal{L}u^n(r,x)+f\left(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)\right)\right)dr\\ \qquad+\displaystyle\int_s^T g\left(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)\right)d\overleftarrow{B}_r,\qquad t\leq s\leq T.\end{cases}$$

We need the following results.

Proposition 5.1. *Under the assumptions (A). Let* $\left(X^{t,x}\right)$ *be the unique solution of SDEs (5.1) and for a fixed* $n \in \mathbb{N}^*$*, let* $\left(Y^{n,t,x}, Z^{n,t,x}\right)$ *be the unique solution of the BDSDEs*

$$\begin{aligned}Y_s^{n,t,x} &= h\left(X_T^{t,x}\right)+\int_s^T f\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right)dr\\ &\quad+\int_s^T g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right)d\overleftarrow{B}_r-\int_t^T Z_r^{n,t,x}dW_r.\end{aligned} \tag{5.7}$$

Then for any $s \in [t,T]$

$$Y_r^{n,s,X_s^{t,x}}=Y_r^{n,t,x},\qquad Z_r^{n,s,X_s^{t,x}}=Z_r^{n,t,x},\quad\text{for a.e. }r\in[s,T],\ x\in\mathbb{R}^d.$$

Proof: The proof is similar to the proof of Proposition 3.4 in Q. Zhang and H. Zhao [10] . ■

Using Proposition 3.1, by the same computation as in (5.5) , we have that the sequence $\left(Y_s^{t,x,n}, Z_s^{t,x,n}\right)$ are bounded in $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$, i.e.,

$$\sup_n\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x,n}\right|^2+\int_t^T\left|Z_s^{t,x,n}\right|^2 ds\right)<\infty. \tag{5.8}$$

Also by Proposition 3.1 applying with $k(\cdot) = \psi(\cdot)$, $\sigma_t = 0$ $\eta_t = 0$, $\lambda = c$ and $\gamma = \alpha$, we can proof by the same computation as in Theorem 4.1, that $\left(Y_s^{t,x,n}, Z_s^{t,x,n}\right)_{s\in[0,T]}$ is a Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$, i.e., there exists a $\left(Y_s^{t,x},Z_s^{t,x}\right)_{s\in[0,T]}\in\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ such that

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x,n}-Y_s^{t,x}\right|^2+\int_t^T\left|Z_s^{t,x,n}-Z_s^{t,x}\right|^2ds\right)\to 0,\quad \text{as } n\to\infty. \tag{5.9}$$

Under the assumptions (A) if we define $u^n(t,x)=Y_t^{n,t,x}$ and $\sigma^*\nabla u^n(t,x)=Z_t^{n,t,x}$. Then by a direct application of Proposition 5.1, and Fubini's Theorem, we have

$$u^n\left(s,X_s^{t,x}\right)=Y_s^{n,t,x},\qquad \sigma^*\nabla u^n\left(s,X_s^{t,x}\right)=Z_s^{n,t,x},\quad for\ a.e.\ s\in[t,T],x\in\mathbb{R}^d. \tag{5.10}$$

Theorem 5.2. *Under hypothesis (A), if we define $u^n(s,x)=Y_s^{n,t,x}$. Then the SPDEs $\left(\mathcal{P}^{(f,g,u^n)}\right)$ admits a unique Sobolev solution u^n, where $\left(Y_s^{n,t,x},Z_s^{n,t,x}\right)_{s\in[t,T]}$ is the unique solution of the BDSDEs* (5.7) *and*

$$u^n\left(s,X_s^{t,x}\right)=Y_s^{n,t,x}\quad and\quad \sigma^*\nabla u^n\left(s,X_s^{t,x}\right)=Z_s^{n,t,x},\quad for\ (s,\omega,x)\ in\ [t,T]\times\Omega\times\mathbb{R}^d. \tag{5.11}$$

Proof: Existence. For each $(s,x)\in[t,T]\otimes\mathbb{R}^d$, define $u^n(s,x)=Y_s^{n,t,x}$ and $\sigma^*\nabla u^n(s,x)=Z_s^{n,t,x}$, where $\left(Y_s^{n,t,x},Z_s^{n,t,x}\right)\in\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ is the solution of Eq (5.7). Then by (5.10)

$$u^n\left(s,X_s^{t,x}\right)=Y_s^{n,t,x},\qquad \sigma^*\nabla u^n\left(s,X_s^{t,x}\right)=Z_s^{n,t,x},\qquad for\ a.e.\ s\in[t,T],\ x\in\mathbb{R}^d.$$

Set

$$\begin{cases} F^n(s,x)=f\left(s,x,u^n(s,x),\sigma^*\nabla u^n(s,x)\right),\\ G^n(s,x)=g\left(s,x,u^n(t,x),\sigma^*\nabla u^n(s,x)\right).\end{cases}$$

Then $\left(Y_s^{n,t,x},Z_s^{n,t,x}\right)\in\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ solve

$$Y_s^{n,t,x}=h\left(X_T^{t,x}\right)+\int_s^T F^n\left(r,X_r^{t,x}\right)dr+\int_s^T G^n\left(r,X_r^{t,x}\right)d\overleftarrow{B}_r-\int_t^T Z_r^{n,t,x}dW_r.$$

Moreover, by Lemma 5.1 and (5.8), we have

$$\begin{aligned}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left(\left|u^n(s,x)\right|^2+\left|\sigma^*\nabla u^n(s,x)\right|^2\right)ds\pi(x)\,dx\right) &\leq \frac{1}{k_1}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left(\left|Y_s^{n,t,x}\right|^2+\left|Z_s^{n,t,x}\right|^2\right)ds\pi(x)\,dx\right),\\ &< \infty.\end{aligned}$$

From $(A.3)\,(i)$ and $(A.4)$, we have

$$\begin{aligned}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left|F^n(s,x)\right|^2ds\ \pi(x)\,dx\right) &\leq 2\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left(c\left|\sigma^*\nabla u^n(s,x)\right|^2+\left|f(s,x,0,0)\right|^2+\varphi\left(\left|u^n(s,x)\right|\right)^2\right)ds\pi(x)\,dx\right)\\ &< \infty.\end{aligned}$$

And from $(A.3)\,(ii)$, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left|G^n(s,x)\right|^2ds\pi(x)\,dx\right)<\infty.$$

Using a some ideas as in the proof of Theorem 2.1 in [5] similar to the argument as in section 4 in [10], we know that $u^n(t,x)$ is the Sobolev solution of the following SPDE:

$$\begin{cases} u^n(t,x)=h(x)+\int_s^T\left(\mathcal{L}u^n(r,x)+F^n(r,x)\right)dr\\ \qquad +\int_s^T G^n(r,x)\,d\overleftarrow{B}_r,\qquad t\leq s\leq T.\end{cases} \tag{5.12}$$

Noting that by the definition of $F^n(r,x)$ and $G^n(r,x)$, from (5.11), we have that u^n is the Sobolev solution of Eq $\left(\mathcal{P}^{(f,g,u^n)}\right)$.

Uniqueness: Let u^n be a solution of Eq $\left(\mathcal{P}^{(f,g,u^n)}\right)$. Define the same notation in the existence part for F^n and G^n, since u^n is a solution, so $E\left(\int_{\mathbb{R}^d}\int_t^T\left(|u^n(s,x)|^2+|\sigma^*\nabla u^n(s,x)|^2\right)ds\pi(x)\,dx\right)<\infty$. From a similar computation as in existence part, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T\left(|F^n(s,x)|^2+|G^n(s,x)|^2\right)ds\;\pi(x)\,dx\right)<\infty.$$

Then, for (5.11) it follows from Proposition 2.3 in [3] that, for and $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, a.e. $s\in[t,T]$, a.s.

$$\begin{aligned}\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\,d\phi_t(r,x)dx+\int_{\mathbb{R}^d}u^n(r,x)\,\phi_t(r,x)\,dx &- \int_{\mathbb{R}^d}h(x)\,\phi_t(T,x)\,dx-\int_s^T\int_{\mathbb{R}^d}u^n(r,x)\,\mathcal{L}^*\phi_t(r,x)\,drdx\\ &= \int_{\mathbb{R}^d}\int_s^T F^n(r,x)\,\phi_t(r,x)\,drdx\\ &+ \int_{\mathbb{R}^d}\int_s^T G^n(r,x)\,\phi_t(r,x)\,d\overleftarrow{B}_r dx.\end{aligned}$$

Now using $\phi_t(r,x)=\phi\left(\hat{X}_r^{t,x}\right)J\left(\hat{X}_r^{t,x}\right)$ and by a change of variable, we get

$$\begin{aligned}\int_{\mathbb{R}^d}u^n(r,x)\,\phi_t(r,x)dx &= \int_{\mathbb{R}^d}u^n\left(r,X_r^{t,x}\right)\phi(x)\,dx,\\ \int_{\mathbb{R}^d}h(x)\,\phi_t(T,x)\,dx &= \int_{\mathbb{R}^d}h\left(X_r^{t,x}\right)\phi(x)\,dx,\\ \int_{\mathbb{R}^d}\int_s^T F^n(r,x)\,\phi_t(r,x)\,drdx &= \int_{\mathbb{R}^d}\int_s^T F^n\left(s,X_r^{t,x}\right)\phi(x)\,drdx,\\ \int_{\mathbb{R}^d}\int_s^T G^n(r,x)\,\phi_t(r,x)\,d\overleftarrow{B}_r dx &= \int_{\mathbb{R}^d}\int_s^T G^n\left(s,X_r^{t,x}\right)\phi(x)\,d\overleftarrow{B}_r dx,\end{aligned}$$

by a change of variable $y=X_r^{t,x}$ and integration by part formula, we obtain

$$\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\,d\phi_t(r,x)\,dx=\int_{\mathbb{R}^d}\int_s^T(\sigma^*\nabla u^n)\left(r,X_r^{t,x}\right)\phi(x)\,dW_r dx+\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\,\mathcal{L}^*\phi_t(r,x)\,drdx.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^d}u^n\left(r,X_r^{t,x}\right)\phi(x)dx &= \int_{\mathbb{R}^d}h\left(X_r^{t,x}\right)\phi(x)\,dx+\int_{\mathbb{R}^d}\int_s^T F^n\left(s,X_r^{t,x}\right)\phi(x)\,drdx\\ &+\int_{\mathbb{R}^d}\int_s^T G^n\left(s,X_r^{t,x}\right)\phi(x)\,d\overleftarrow{B}_r dx-\int_{\mathbb{R}^d}\int_s^T(\sigma^*\nabla u^n)\left(r,X_r^{t,x}\right)\phi(x)\,dW_r dx.\end{aligned}$$

From the arbitrariness of ϕ we know that $\left\{u^n\left(r,X_r^{t,x}\right),(\sigma^*\nabla u^n)\left(r,X_r^{t,x}\right),\ t\leq r\leq T\right\}$ is a solution of the following BDSDE

$$Y_s^{n,t,x}=h(X_T^{t,x})+\int_s^T F^n\left(r,X_r^{t,x}\right)dr+\int_s^T G^n\left(r,X_r^{t,x}\right)d\overleftarrow{B}_r-\int_t^T Z_r^{n,t,x}dW_r,\ t\leq s\leq T.$$

Then from the definitions of F^n and G^n it follows that $\left\{u^n\left(r,X_r^{t,x}\right),(\sigma^*\nabla u^n)\left(r,X_r^{t,x}\right),\ t\leq r\leq T\right\}$ solve BDSDE (5.7).
If there is another solution $\tilde{u}^n$ to Eq. $\left(\mathcal{P}^{(f,g,u^n)}\right)$, then by the same procedure, we can find another solution $\left(\tilde{Y}_s^{t,x,n},\tilde{Z}_s^{t,x,n}\right)$ solve the BDSDE (5.7), where

$$\tilde{u}^n\left(s,X_s^{t,x}\right)=\tilde{Y}_s^{n,t,x},\qquad\sigma^*\nabla\tilde{u}^n\left(s,X_s^{t,x}\right)=\tilde{Z}_s^{n,t,x},\quad for\ a.e.s\in[t,T],\ x\in\mathbb{R}^d.$$

By Theorem 4.1, the solution of Eq. (5.7) is unique, therefore

$$\tilde{Y}_s^{n,t,x}=Y_s^{n,t,x},\qquad for\ a.e.s\in[t,T],\ x\in\mathbb{R}^d.$$

Now, applying Lemma 5.1 again, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T|\tilde{u}^n(s,x)-u^n(s,x)|^2ds\pi(x)\,dx\right)\leq\frac{1}{k_1}\left(\int_{\mathbb{R}^d}\int_t^T\left|\tilde{Y}_s^{n,t,x}-Y_s^{n,t,x}\right|^2ds\pi(x)\,dx\right)=0.$$

So $\tilde{u}^n(s,x)=u^n(s,x)$, for a.e. $s\in[0,T]$, $x\in\mathbb{R}^d$ a.s..Uniqueness is proved. ∎

Proposition 5.2. *Under assumptions* (A), *let* $\left(Y_t^{t,x}, Z_t^{t,x}\right)$ *be the solution of Eq.* (5.3). *If we define* $u(s,x) = Y_s^{t,x}$, *then* $\sigma^*\nabla u(s,x)$ *exists for a.e.* $s \in [t,T]$, $x \in \mathbb{R}^d$ *a.s., and*

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \qquad \sigma^*\nabla u(s, X_s^{t,x}) = Z_s^{t,x}, \quad for\ a.e.\ s \in [t,T],\ x \in \mathbb{R}^d. \tag{5.13}$$

Proof: See Proposition 4.2 in Q. Zhang, and H. Zhao [10] ■

In the rest part of this section, we study Eq $\left(\mathcal{P}^{(f,g)}\right)$. Then by Theorem 5.2, Proposition 5.2, Lemma 5.1 and estimation (5.9), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T \left(|u^n(s,x) - u(s,x)|^2 + |\sigma^*\nabla u^n(s,x) - \sigma^*\nabla u(s,x)|^2\right) ds\pi(x)\,dx \\
\leq\ & \frac{1}{k_1}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(\left|u^n\left(s, X_s^{t,x}\right) - u\left(s, X_s^{t,x}\right)\right|^2 + \left|\sigma^*\nabla u^n\left(s, X_s^{t,x}\right) - \sigma^*\nabla u\left(s, X_s^{t,x}\right)\right|^2\right) ds\pi(x)\,dx\right), \\
\to\ & 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{5.14}$$

With (5.14) we prove the Theorem 5.1 in this section.

Proof of Theorem 5.1: Existence, by Lemma 5.1 and (5.13), we see that

$$\sigma^*\nabla u(t,x) = Z_t^{t,x}, \qquad for\ a.e.\ t \in [0,T],\ x \in \mathbb{R}^d.$$

Also, by Lemma 5.1 and (5.5), we have

$$\begin{aligned}
\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(|u(s,x)|^2 + |\sigma^*\nabla u(s,x)|^2\right) ds\pi(x)\,dx\right) &\leq \frac{1}{k_1}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(\left|Y_s^{t,x}\right|^2 + \left|Z_s^{t,x}\right|^2\right) ds\pi(x)\,dx\right), \\
&< \infty.
\end{aligned}$$

Now we will prove that u satisfies the definition 5.1. Let $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^d)$, since for any n, u^n is a Sobolev solution to the problem $(P^{(f,g,u^n)})$, we then have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)\,drdx + \int_{\mathbb{R}^d} u^n(r,x)\varphi(r,x)\,dx - \int_{\mathbb{R}^d} h(x)\varphi(T,x)\,dx \\
&- \frac{1}{2}\int_{\mathbb{R}^d}\int_s^T \sigma^* u^n(r,x)\sigma^*\varphi(r,x)\,drdx - \int_{\mathbb{R}^d}\int_s^T u^n div\left((b-A)\varphi\right)(r,x)\,drdx \\
&= \int_{\mathbb{R}^d}\int_s^T f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\varphi(r,x)\,drdx + \int_{\mathbb{R}^d}\int_s^T g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\varphi(r,x)\,d\overleftarrow{B}_r dx,
\end{aligned} \tag{5.15}$$

By proving that along a subsequence (5.15) converges to (5.2) in $\mathbb{L}^2(\Omega)$, we have that $u(t,x)$ satisfies (5.2). We only need to show that along a subsequence as $n \to \infty$

$$\begin{cases}
\int_{\mathbb{R}^d}\int_s^T \left(f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right)\varphi(r,x)\,drdx \to 0, \\
\int_{\mathbb{R}^d}\int_s^T \left(g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - g(r,x,u(r,x),\sigma^*\nabla u(r,x))\right)\varphi(r,x)\,d\overleftarrow{B}_r dx \to 0.
\end{cases}$$

Firstly. Since $\varphi \in C_c^\infty$ then φ is belong in $\mathbb{L}^2\left(\mathbb{R}^d \times [s,T], dt \otimes dx\right)$ and by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_s^T \left(f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right)\varphi(r,x)\,drdx\right|^2 \\
\leq\ & \int_{\mathbb{R}^d}\int_s^T \left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2 \pi(x)\,drdx \int_{\mathbb{R}^d}\int_s^T \frac{|\varphi(r,x)|^2}{\pi(x)}drdx, \\
\leq\ & C\int_{\mathbb{R}^d}\int_s^T \left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2 \pi(x)\,drdx.
\end{aligned}$$

Also we have by Lemma 5.1, and by definition of $u^n(r, X_r^{s,x})$, $\sigma^*\nabla u^n(r, X_r^{s,x})$ that,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T \left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)) - f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2 dr\pi(x)\,dx, \\
\leq\ & \frac{1}{k}\mathbb{E}\int_{\mathbb{R}^d}\int_s^T \left|f(r,x,Y_r^{n,s,x},Z_r^{n,s,x}) - f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2 dr\pi(x)\,dx,
\end{aligned}$$

using $(A.3)\,(i)$ and $(a+b)^2 \le 2a^2+2b^2$, we have

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T |f(r,x,Y_r^{n,s,x},Z_r^{n,s,x}) \quad -f(r,x,Y_r^{s,x},Z_r^{s,x})|^2 dr\pi(x)\,dx \le 2c\mathbb{E}\int_{\mathbb{R}^d}\int_s^T |Z_r^{n,s,x}-Z_r^{s,x}|^2 dr\pi(x)\,dx$$
$$+2\mathbb{E}\int_{\mathbb{R}^d}\int_s^T |f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})|^2 dr\pi(x)\,dx.$$

We only need to prove that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T \left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2 dr\pi(x)\,dx \to 0, \quad as\ n\to\infty.$$

Applying assumption $(A.1)$, we have

$$\lim_{n\to\infty}\left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2=0.$$

Since $\mathbb{E}\int_{\mathbb{R}^d}\int_t^T \left|Z_s^{t,x,n}\right|^2 ds\pi(x)\,dx < \infty$, then there exists a subsequence which we still denote $Z^{t,x,n}\to Z^{s,x}$ such that $\mathbb{E}\int_{\mathbb{R}^d}\int_t^T \left|Z_s^{t,x}\right|^2 ds\pi(x)\,dx<\infty$, using (5.8), $(A.3)\,(i)$ and $(A.4)$, we have

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T |f(r,x,Y_r^{n,s,x},Z_r^{s,x})|^2 dr\pi(x)\,dx \le \mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left(c\,|Z_r^{s,x}|^2+|f(r,x,0,0)|^2+\varphi\left(\sup_{t\le r\le T}|Y_r^{n,s,x}|\right)^2\right)dr\pi(x)\,dx,$$
$$< \infty.$$

According to the Lebesgue's dominated convergence Theorem, it follows that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T \left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2 dr\pi(x)\,dx \to 0, \quad as\ n\to\infty,$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_s^T f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\,\varphi(r,x)drdx=\int_{\mathbb{R}^d}\int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x))\,\varphi(r,x)\,drdx.$$

Secondly It remains to prove that

$$\int_{\mathbb{R}^d}\int_s^T g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\,\varphi(r,x)\,d\overleftarrow{B}_r dx,$$

tends to

$$\int_{\mathbb{R}^d}\int_s^T g(r,x,u(r,x),\sigma^*\nabla u(r,x))\,\varphi(r,x)\,d\overleftarrow{B}_r dx,$$

as n tends to ∞. Arguing as in the proof of Theorem 4.1, we get the following limit in probability as $n\to\infty$, $\int_0^T g\left(r,X_r^{t,x},u^n\left(r,X_r^{t,x}\right),\sigma^*\nabla u^n\left(r,X_r^{t,x}\right)\right)d\overleftarrow{B}_r \to \int_0^T g\left(r,X_r^{t,x},u\left(s,X_r^{t,x}\right),\sigma^*\nabla u\left(r,X_r^{t,x}\right)\right)$ $d\overleftarrow{B}_r$.

By Lemma 5.1, (5.5) and (5.8), we have

$$\int_{\mathbb{R}^d}\left|\int_s^T (g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-g(r,x,u(r,x),\sigma^*\nabla u(r,x)))\varphi(r,x)\,\pi(x)\,d\overleftarrow{B}_r\right|\pi^{-1}(x)\,dx<\infty,$$

i.e $\int_s^T (g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-g(r,x,u(r,x),\sigma^*\nabla u(r,x)))\varphi(r,x)\,\pi(x)\,d\overleftarrow{B}_r$ belongs to $\mathbb{L}^1\left(\mathbb{R}^d,\pi^{-1}(x)\,dx\right)$.

Hence, using Lemma 5.1 we get, for every $s\in[0,T]$

$$\int_{\mathbb{R}^d}\left|\int_s^T (g(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-g(r,x,u(r,x),\sigma^*\nabla u(r,x)))\varphi(r,x)\,\pi(x)\,d\overleftarrow{B}_r\right|\pi^{-1}(x)\,dx$$
$$\le \frac{1}{k_1}\int_{\mathbb{R}^d}\mathbb{E}\left|\int_s^T\left(g\left(r,X_r^{t,x},u^n\left(r,X_r^{t,x}\right),\sigma^*\nabla u^n\left(r,X_r^{t,x}\right)\right)-g\left(r,X_r^{t,x},u\left(r,X_r^{t,x}\right),\sigma^*\nabla u\left(r,X_r^{t,x}\right)\right)\right)\varphi\left(r,X_r^{t,x}\right)\right|$$
$$\pi\left(X_r^{t,x}\right)d\overleftarrow{B}_r\pi^{-1}(x)\,dx$$
$$=\frac{1}{k_1}\int_{\mathbb{R}^d}\mathbb{E}\int_s^T\left(g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right)-g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\right)\varphi\left(r,X_r^{t,x}\right)\pi\left(X_r^{t,x}\right)d\overleftarrow{B}_r\pi^{-1}(x)\,dx.$$

Since

$$\begin{cases} \sup_n \mathbb{E}\int_s^T \left(g\left(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}\right) - g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\right)\varphi(r, X_r^{t,x})\pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r < \infty, \\ and \\ \int_s^T \left(g\left(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}\right) - g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\right)\varphi\left(r, X_r^{t,x}\right)\pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r \text{ converges to 0 in probability,} \end{cases}$$

it follows according to the Lebesgue's dominated convergence theorem that

$$\lim_n \mathbb{E}\int_s^T \left(g\left(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}\right) - g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\right)\varphi\left(r, X_r^{t,x}\right)\pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r = 0.$$

Therefore $u(t,x)$ satisfies (5.2), i.e. it is a Sobolev solution of $(\mathcal{P}^{(f,g)})$. Theorem 5.1. is proved. ■

6 Conclusion

In this paper we studied the BDSDEs and SPDEs. We introduced a BDSDE with weak monotonicity and general growth generators and a square integrable terminal datum. We studied the relationship between BDSDEs and SPDEs in this case, and we give the Sobolev solutions to some semilinear stochastic partial differential equations (SPDEs) with a general growth and a weak monotonicity generators. By probabilistic solution.

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Competing Interests

Authors have declared that no competing interests exist.

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