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Bayesian and Maximum Likelihood Estimation of the Shape Parameter of Exponential Inverse Exponential Distribution: A Comparative Approach

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Authors' contributions

This work was carried out in collaboration among all authors. Author IBE designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors FBM and AAS managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

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Abstract

This paper aims at making Bayesian analysis on the shape parameter of the exponential inverse exponential distribution using informative and non-informative priors. Bayesian estimation was carried out through a Monte Carlo study under 10,000 replications. To assess the effects of the assumed prior distributions and loss function on the Bayesian estimators, the mean square error has been used as a criterion. Overall, simulation results indicate that Bayesian estimation under QLF outperforms the maximum likelihood estimation and Bayesian estimation under alternative loss functions irrespective of the nature of the prior and the sample size. Also, for large sample sizes, all methods perform equally well.

Keywords: Exponential inverse exponential distribution; Bayesian analysis; prior distributions; loss functions; mean square error.

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1 Introduction

Some standard probability distributions have been used over the years for modeling real life datasets. However, research has shown that most of these distributions do not adequately describe heavily slewed datasets, which limits their applicability in many real-world situations. Recently, numerous extended or compound probability distributions have been proposed in the literature for modeling real life situations. These compound distributions are found to be skewed, flexible and much better in statistical modeling compared to their standard counterparts [1-14].

Due to the abovementioned facts, [15] developed an exponential inverse exponential distribution (EIED) with two parameters (a shape and scale parameter). This distribution has been found to be skewed and flexible with an increasing hazard rate and different shapes and also performed better than the exponential distribution based on applications of the models to three lifetime datasets [15].

In [15], the probability density function (pdf), the cumulative distribution function (cdf), the survival function (sf), the hazard function (or failure rate) and quantile function (qf) of the EIED are respectively defined as:

$$f(x)=\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2}e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \tag{1.1}$$

$$F(x)=1-e^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \tag{1.2}$$

$$S(x)=1-F\left(x\right)=\mathrm{e}^{-\alpha\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)} \tag{1.3}$$

$$h(x)=\frac{f\left(x\right)}{S\left(x\right)}=\frac{\alpha\theta}{x^2}\frac{e^{-\frac{\theta}{x}}}{\left[1-e^{-\frac{\theta}{x}}\right]^2} \tag{1.4}$$

and

$$q\left(u\right)=\theta\left\{\log\left[1-\alpha\left(\log\left(1-u\right)\right)^{-1}\right]\right\}^{-1} \tag{1.5}$$

For $x > 0, \alpha, \theta > 0$ where u is a uniform variate on the interval $0 \le u \le 1$, α is a shape parameter and θ is a scale parameter of the exponential inverse exponential distribution (EIED).

A graphical representation of the above functions using some arbitrary parameter values is displayed in the following figures:

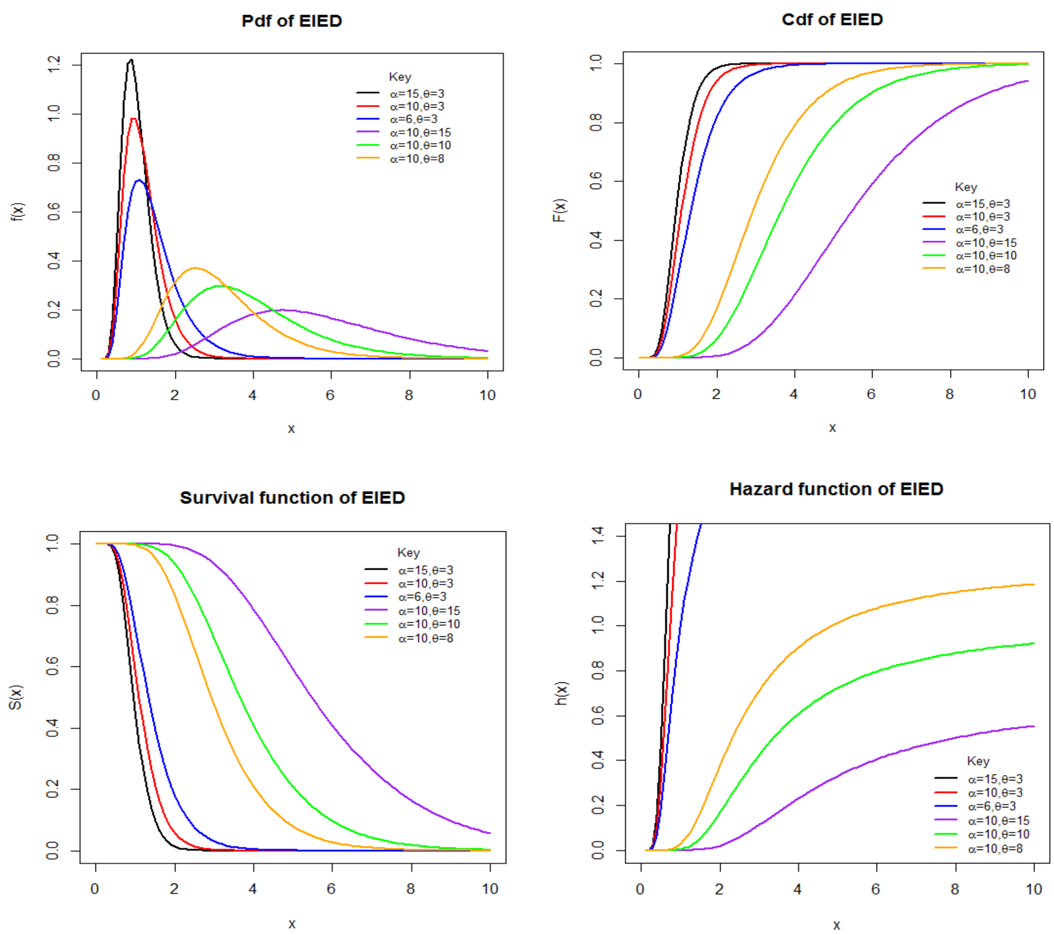

Fig. 1. Plots of the PDF, CDF, survival function and hazard function of the EIED for selected parameter values

More about the important mathematical and statistical properties, maximum likelihood estimation of parameters and applications of the Exponential Inverse Exponential distribution showing its efficiency over Inverse Exponential distribution using real life datasets can be found in [15].

There are two basic approaches to parameter estimation and these are the frequentist (or classical) and the Bayesian approaches (or non-classical methods). The classical theory of estimation involves a situation where the parameters are considered to be constant but unknown whereas the parameters are considered to be unknown and random just like variables under non classical approach. The most widely used method in classical theory is the method of maximum likelihood estimation while the Bayesian estimation method is used in the non classical theory. However, in most real life problems described by life time distributions, the parameters cannot be considered as constants in all the life testing period [16-18]. Following this narrative, it becomes obvious that the classical (frequentist) approach can no longer handle adequately problems of parameter estimation in life time models and therefore the need for non classical or Bayesian estimation in life time models.

Estimation of parameters in a distribution differs by method from one parameter of the distribution to another and therefore this study aims at estimating the shape parameter of the EIED using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation.

The aim of this article is to estimate the shape parameter of the EIED using Bayesian approach assuming uniform prior, Jeffrey's prior and gamma prior distributions with three loss functions. The rest of this paper is organized as follows: in Section 2, maximum likelihood estimator (MLE) for the shape parameter is obtained. In Section 3, Bayesian estimators based on the different loss functions by assuming uniform, Jeffrey's and gamma prior distributions are derived. The proposed estimators are compared in relation of their mean squared error (MSE) in Section 4. Finally, the conclusion is provided in Section 5.

2 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from a population *X* of size 'n' independently and identically distributed random variables with probability density function $f(x)$. The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Given that the values, $\underline{x} = (x_1, x_2, ..., x_n)$ are obtained independently from an EIED with unknown parameters *α* and *Θ*, the likelihood function is given by:

$$L(\underline{x} \mid \alpha, \theta) = P(x_1, x_2, ..., x_n \mid \alpha, \theta) = \prod_{i=1}^{n} P(\underline{x} \mid \alpha, \theta) \tag{2.1}$$

The likelihood function, $L(\underline{x} \mid \alpha, \theta)$ based on the pdf of EIED is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L(\underline{x} \mid \alpha, \theta) = (\alpha\theta)^n \frac{\mathrm{e}^{-\theta\sum_{i=1}^{n}\frac{1}{x_i}}}{\prod_{i=1}^{n}\left(x_i^2\left[1-\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2\right)} \mathrm{e}^{-\alpha\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{2.2}$$

For the shape parameter of the EIED α, the likelihood function is given by;

$$L(\underline{x} \mid \alpha) \propto \alpha^n \mathrm{e}^{-\alpha\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)}$$

$$L(\underline{x} \mid \alpha) = K\alpha^n \mathrm{e}^{-\alpha\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{2.3}$$

Where $K = \dfrac{\theta^n \mathrm{e}^{-\theta\sum_{i=1}^{n}\frac{1}{x_i}}}{\prod_{i=1}^{n}\left(x_i^2\left[1-\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2\right)}$ is a constant which is independent of the shape parameter, α .

Let $l = \log L(\underline{x} \mid \alpha)$ denote the log-likelihood function such that

$$l = n \log \alpha - \alpha \sum_{i=1}^{n} \left(\frac{e^{-\frac{\theta}{x_i}}}{1 - e^{-\frac{\theta}{x_i}}} \right) \tag{2.4}$$

Differentiating l with respect to α and setting the derivative equal to zero gives;

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\frac{e^{-\frac{\theta}{x_i}}}{1 - e^{-\frac{\theta}{x_i}}} \right) = 0 \tag{2.5}$$

Solving equation (2.5) for α yields the maximum likelihood estimator (MLE) $\hat{\alpha}$ as:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \left(\frac{e^{-\frac{\theta}{x_i}}}{1 - e^{-\frac{\theta}{x_i}}} \right)} \tag{2.6}$$

Details concerning the maximum likelihood estimation of the scale parameter of the EIED can be found in [15].

3 Bayesian Estimation

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s).

In this study, two non-informative priors (uniform and Jeffrey) and an informative prior (gamma) will be considered for estimating the shape parameter of the EIED. These assumed prior distributions have been used widely by several authors including [19-27]. This study also considers three loss functions including square error, quadratic and precautionary loss functions which have also been used previously by some researchers such as [28-38] etc. The study also considered deriving the estimators of the shape parameter in closed-form using the Bayesian approach because of the usefulness of Closed-form estimators as recently demonstrated by [39] and [40]. The stated prior distributions and loss functions are defined as follows:

a. The uniform prior is defined as:

$$p(\alpha) \propto 1; 0 < \alpha < \infty \tag{3.1}$$

b. Also, the Jeffrey's prior is defined as:

$$p(\alpha) \propto \frac{1}{\alpha}; 0 < \alpha < \infty \tag{3.2}$$

c. Also, the gamma prior is defined as:

$$P(\alpha)=\frac{a^b}{\Gamma(b)}\alpha^{b-1}e^{-a\alpha} \tag{3.3}$$

i. Squared Error Loss Function (*SELF*)

The squared error loss function relating to the shape parameter α is defined as:

$$L(\alpha,\alpha_{SELF})=(\alpha-\alpha_{SELF})^2 \tag{3.4}$$

where α_{SELF} is the estimator of the parameter α under *SELF*.

ii. Quadratic Loss Function (QLF)

The quadratic loss function is defined from [41] as

$$L(\alpha,\alpha_{QLF})=\left(\frac{\alpha-\alpha_{QLF}}{\alpha}\right)^2 \tag{3.5}$$

where α_{QLF} is the estimator of the parameter α under *QLF*.

iii. Precautionary Loss Function (*PLF*)

The precautionary loss function (*PLF*) introduced by [42] is an asymmetric loss function and is defined as

$$L(\alpha_{PLF},\alpha)=\frac{(\alpha_{PLF}-\alpha)^2}{\alpha_{PLF}} \tag{3.6}$$

where α_{PLF} is the estimator of the shape parameter α under *PLF*.

The posterior distribution of a parameter is the distribution of the parameter after observing the available data and it is obtained by using Bayes' theorem in relation to the shape parameter α, likelihood function and prior distribution as follows:

$$P(\alpha\,|\,\underline{x})=\frac{P(\alpha,\underline{x})}{P(\underline{x})}=\frac{P(\underline{x}\,|\,\alpha)P(\alpha)}{P(\underline{x})}=\frac{P(\underline{x}\,|\,\alpha)P(\alpha)}{\int P(\underline{x}\,|\,\alpha)P(\alpha)d\alpha}=\frac{L(\underline{x}\,|\,\alpha)P(\alpha)}{\int L(\underline{x}\,|\,\alpha)P(\alpha)d\alpha} \tag{3.7}$$

where $P(\underline{x})$ is the marginal distribution of X and $P(\underline{x})=\sum_{x}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)$ when the prior distribution of α is discrete and $P(\underline{x})=\int_{-\infty}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)d\alpha$ when the prior distribution of α is continuous. Also note that $p(\alpha)$ and $L(\underline{x}\,|\,\alpha)$ are the prior distribution and the Likelihood function respectively.

3.1 Bayesian analysis under uniform prior with three loss functions

The posterior distribution of the shape parameter α assuming a uniform prior distribution is obtained from (3.7) using integration by substitution method as:

$$P(\alpha \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)\right)^{n+1}}{\Gamma(n+1)\alpha^{-n} e^{\alpha \sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}} \tag{3.8}$$

Bayes estimators under uniform prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF} = E(\alpha \mid \underline{x}) = \int_0^\infty \alpha P(\alpha \mid \underline{x})\, d\alpha = \frac{n+1}{\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)} \tag{3.9}$$

$$\alpha_{QLF} = \frac{E(\alpha^{-1} \mid \underline{x})}{E(\alpha^{-2} \mid \underline{x})} = \frac{\int_0^\infty \alpha^{-1} P(\alpha \mid \underline{x})\, d\alpha}{\int_0^\infty \alpha^{-2} P(\alpha \mid \underline{x})\, d\alpha} = \frac{(n-1)}{\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)} \tag{3.10}$$

and

$$\alpha_{PLF} = \left\{E(\alpha^2 \mid \underline{x})\right\}^{\frac{1}{2}} = \left\{\int_0^\infty \alpha^2 P(\alpha \mid \underline{x})\, d\alpha\right\}^{\frac{1}{2}} = \frac{\left[(n+1)(n+2)\right]^{0.5}}{\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)} \tag{3.11}$$

3.2 Bayesian analysis under Jeffrey’s prior with three loss functions

The posterior distribution of the shape parameter α for a given data assuming a Jeffrey’s prior distribution is obtained from (3.7) using integration by substitution method as:

$$P(\alpha \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)\right)^{n}}{\Gamma(n)\alpha^{-n-1} e^{\alpha \sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right)}} \tag{3.12}$$

Bayes estimators under Jeffrey's prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF}=E\left(\alpha\mid\underline{x}\right)=\int_{0}^{\infty}\alpha P\left(\alpha\mid\underline{x}\right)d\alpha=\frac{n}{\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.13}$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\mid\underline{x}\right)}{E\left(\alpha^{-2}\mid\underline{x}\right)}=\frac{\int_{0}^{\infty}\alpha^{-1}P\left(\alpha\mid\underline{x}\right)d\alpha}{\int_{0}^{\infty}\alpha^{-2}P\left(\alpha\mid\underline{x}\right)d\alpha}=\frac{\left(n-2\right)}{\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.14}$$

and

$$\alpha_{PLF}=\left\{E\left(\alpha^{2}\mid\underline{x}\right)\right\}^{\frac{1}{2}}=\left\{\int_{0}^{\infty}\alpha^{2}P\left(\alpha\mid\underline{x}\right)d\alpha\right\}^{\frac{1}{2}}=\frac{\left[n\left(n+1\right)\right]^{0.5}}{\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.15}$$

3.3 Bayesian analysis under gamma prior with three loss functions

The posterior distribution of the shape parameter α for a given data assuming a gamma prior distribution is obtained from (3.7) using integration by substitution method as

$$P\left(\alpha\mid\underline{x}\right)=\frac{\left(a+\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)\right)^{n+b}}{\Gamma\left(n+b\right)\alpha^{-\left(n+b-1\right)}e^{\alpha\left(a+\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)\right)}} \tag{3.16}$$

Bayes estimators under gamma prior with *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF}=E\left(\alpha\mid\underline{x}\right)=\int_{0}^{\infty}\alpha P\left(\alpha\mid\underline{x}\right)d\alpha=\frac{n+b}{a+\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.17}$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\mid\underline{x}\right)}{E\left(\alpha^{-2}\mid\underline{x}\right)}=\frac{\int_{0}^{\infty}\alpha^{-1}P\left(\alpha\mid\underline{x}\right)d\alpha}{\int_{0}^{\infty}\alpha^{-2}P\left(\alpha\mid\underline{x}\right)d\alpha}=\frac{n+b-2}{a+\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.18}$$

and

$$\alpha_{PLF} = \left\{E\left(\alpha^2 \mid \underline{x}\right)\right\}^{\frac{1}{2}} = \left\{\int_0^\infty \alpha^2 P\left(\alpha \mid \underline{x}\right) d\alpha\right\}^{\frac{1}{2}} = \frac{\left[\left(n+b+1\right)\left(n+b\right)\right]^{0.5}}{a + \sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right)} \tag{3.19}$$

4 Results and Discussion

In this section, Monte Carlo simulation with R software under 10,000 replications is considered to generate random samples of sizes n = (25, 50, 75, 100, 125, 150) from the EIED using the quantile function (inverse transformation method of simulation) under the following combination of parameter values: $\alpha = 1.0, \theta = 1.0, a = 1.0, b = 1.0$, $\alpha = 3.0, \theta = 0.5, a = 0.5, b = 0.5$, $\alpha = 0.5, \theta = 2.5, a = 0.5, b = 0.5$ and $\alpha = 2.5, \theta = 1.0, a = 1.0, b = 1.0$. The following tables present the results of a simulation study by listing the average estimates of the shape parameter with their respective Mean Square Errors (MSEs) under the appropriate estimation methods which include the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under Uniform Jeffrey and gamma priors respectively. The criterion for evaluating the performance of the estimators in this study is the Mean Square Error (MSE): $MSE = \frac{1}{n} E\left(\hat{\alpha} - \alpha\right)^2$.

Table 1. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under uniform prior

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	1.0	1.0	1.0	1.0	1.0393	1.0809	0.9977	1.1014
					(0.0475)	(0.0562)	(0.0424)	(0.0619)
	3.0	0.5	0.5	0.5	3.1178	3.2426	2.9931	3.3043
					(0.4274)	(0.5061)	(0.3812)	(0.5571)
	0.5	2.5	0.5	0.5	0.5196	0.5404	0.4989	0.5507
					(0.0119)	(0.0141)	(0.0106)	(0.0155)
	2.5	1.0	1.0	1.0	2.5982	2.7021	2.4943	2.7536
					(0.2968)	(0.3515)	(0.2647)	(0.3869)
50	1.0	1.0	1.0	1.0	1.0198	1.0401	0.9994	1.0503
					(0.0221)	(0.0242)	(0.0209)	(0.0256)
	3.0	0.5	0.5	0.5	3.0593	3.1204	2.9981	3.1509
					(0.1989)	(0.2178)	(0.1877)	(0.2301)
	0.5	2.5	0.5	0.5	0.5099	0.5201	0.4997	0.5251
					(0.0055)	(0.0061)	(0.0052)	(0.0064)
	2.5	1.0	1.0	1.0	2.5494	2.6004	2.4984	2.6257
					(0.1382)	(0.1513)	(0.1303)	(0.1598)
75	1.0	1.0	1.0	1.0	1.0145	1.0280	1.0009	1.0347
					(0.0143)	(0.0153)	(0.0138)	(0.0159)
	3.0	0.5	0.5	0.5	3.0434	3.0840	3.0028	3.1042
					(0.1291)	(0.1377)	(0.1239)	(0.1432)
	0.5	2.5	0.5	0.5	0.5072	0.5140	0.5005	0.5174
					(0.0036)	(0.0038)	(0.0034)	(0.0040)
	2.5	1.0	1.0	1.0	2.5362	2.5700	2.5023	2.5868
					(0.0897)	(0.0956)	(0.0860)	(0.0994)

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
100	1.0	1.0	1.0	1.0	1.0111 (0.0107)	1.0212 (0.0113)	1.0010 (0.0104)	1.0263 (0.0116)
	3.0	0.5	0.5	0.5	3.0334 (0.0965)	3.0637 (0.1013)	3.0031 (0.0934)	3.0789 (0.1044)
	0.5	2.5	0.5	0.5	0.5056 (0.0027)	0.5106 (0.0028)	0.5005 (0.0026)	0.5131 (0.0029)
	2.5	1.0	1.0	1.0	2.5278 (0.0670)	2.5531 (0.0704)	2.5025 (0.0649)	2.5657 (0.0725)
125	1.0	1.0	1.0	1.0	1.0081 (0.0081)	1.0162 (0.0085)	1.0000 (0.0079)	1.0202 (0.0087)
	3.0	0.5	0.5	0.5	3.0243 (0.0733)	3.0485 (0.0762)	3.0001 (0.0715)	3.0606 (0.0781)
	0.5	2.5	0.5	0.5	0.5041 (0.0020)	0.5081 (0.0021)	0.500 (0.002)	0.5101 (0.0022)
	2.5	1.0	1.0	1.0	2.5203 (0.0509)	2.5404 (0.0529)	2.5001 (0.0497)	2.5505 (0.0543)
150	1.0	1.0	1.0	1.0	1.007 (0.0070)	1.0138 (0.0072)	1.0004 (0.0068)	1.0172 (0.0073)
	3.0	0.5	0.5	0.5	3.0213 (0.0627)	3.0415 (0.0648)	3.0012 (0.0614)	3.0515 (0.0661)
	0.5	2.5	0.5	0.5	0.5036 (0.0017)	0.5069 (0.0018)	0.5002 (0.0017)	0.5086 (0.0018)
	2.5	1.0	1.0	1.0	2.5178 (0.0435)	2.5346 (0.0450)	2.5010 (0.0426)	2.5429 (0.0459)

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function

Table 2. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under Jeffrey's prior

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	1.0	1.0	1.0	1.0	1.0393 (0.0475)	1.0393 (0.0475)	0.9561 (0.0408)	1.0599 (0.0514)
	3.0	0.5	0.5	0.5	3.1178 (0.4274)	3.1178 (0.4274)	2.8684 (0.3673)	3.1796 (0.4623)
	0.5	2.5	0.5	0.5	0.5196 (0.0119)	0.5196 (0.0119)	0.4781 (0.0102)	0.5299 (0.0128)
	2.5	1.0	1.0	1.0	2.5982 (0.2968)	2.5982 (0.2968)	2.3904 (0.2551)	2.6497 (0.3211)
50	1.0	1.0	1.0	1.0	1.0198 (0.0221)	1.0198 (0.0221)	0.9790 (0.0205)	1.0299 (0.0230)
	3.0	0.5	0.5	0.5	3.0593 (0.1989)	3.0593 (0.1989)	2.9369 (0.1841)	3.0897 (0.2074)
	0.5	2.5	0.5	0.5	0.5099 (0.0055)	0.5099 (0.0055)	0.4895 (0.0051)	0.5149 (0.0058)
	2.5	1.0	1.0	1.0	2.5494 (0.1382)	2.5494 (0.1382)	2.4474 (0.1278)	2.5747 (0.1440)
75	1.0	1.0	1.0	1.0	1.0145 (0.0143)	1.0145 (0.0143)	0.9874 (0.0135)	1.0212 (0.0148)
	3.0	0.5	0.5	0.5	3.0434 (0.1291)	3.0434 (0.1291)	2.9622 (0.1219)	3.0636 (0.1330)

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
	0.5	2.5	0.5	0.5	0.5072 (0.0036)	0.5072 (0.0036)	0.4937 (0.0034)	0.5106 (0.0037)
	2.5	1.0	1.0	1.0	2.5362 (0.0897)	2.5362 (0.0897)	2.4685 (0.0847)	2.5530 (0.0923)
100	1.0	1.0	1.0	1.0	1.0111 (0.0107)	1.0111 (0.0107)	0.9909 (0.0103)	1.0162 (0.0110)
	3.0	0.5	0.5	0.5	3.0334 (0.0965)	3.0334 (0.0965)	2.9727 (0.0923)	3.0485 (0.0986)
	0.5	2.5	0.5	0.5	0.5056 (0.0027)	0.5056 (0.0027)	0.4955 (0.0026)	0.5081 (0.0027)
	2.5	1.0	1.0	1.0	2.5278 (0.0670)	2.5278 (0.0670)	2.4773 (0.0641)	2.5404 (0.0685)
125	1.0	1.0	1.0	1.0	1.0081 (0.0081)	1.0081 (0.0081)	0.9920 (0.0079)	1.0121 (0.0083)
	3.0	0.5	0.5	0.5	3.0243 (0.0733)	3.0243 (0.0733)	2.9759 (0.0710)	3.0364 (0.0746)
	0.5	2.5	0.5	0.5	0.5041 (0.0020)	0.5041 (0.0020)	0.496 (0.002)	0.5061 (0.0021)
	2.5	1.0	1.0	1.0	2.5203 (0.0509)	2.5203 (0.0509)	2.4799 (0.0493)	2.5303 (0.0518)
150	1.0	1.0	1.0	1.0	1.007 (0.0070)	1.0071 (0.0070)	0.9937 (0.0068)	1.0105 (0.0071)
	3.0	0.5	0.5	0.5	3.0213 (0.0627)	3.0213 (0.0627)	2.9811 (0.0609)	3.0314 (0.0636)
	0.5	2.5	0.5	0.5	0.5036 (0.0017)	0.5036 (0.0017)	0.4968 (0.0017)	0.5052 (0.0018)
	2.5	1.0	1.0	1.0	2.5178 (0.0435)	2.5178 (0.0435)	2.4842 (0.0423)	2.5262 (0.0442)

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function

Table 3. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under gamma prior

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	1.0	1.0	1.0	1.0	1.0393 (0.0475)	1.036 (0.043)	0.9563 (0.0374)	1.0558 (0.0464)
	3.0	0.5	0.5	0.5	3.1178 (0.4274)	2.9866 (0.3320)	2.7523 (0.3432)	3.0446 (0.3468)
	0.5	2.5	0.5	0.5	0.5196 (0.0119)	0.5244 (0.0120)	0.4832 (0.0100)	0.5345 (0.0131)
	2.5	1.0	1.0	1.0	2.5982 (0.2968)	2.4390 (0.2073)	2.2514 (0.2353)	2.4854 (0.2116)
50	1.0	1.0	1.0	1.0	1.0198 (0.0221)	1.0189 (0.0211)	0.9790 (0.0196)	1.0289 (0.0220)
	3.0	0.5	0.5	0.5	3.0593 (0.1989)	2.9963 (0.1759)	2.8777 (0.1772)	3.0258 (0.1800)
	0.5	2.5	0.5	0.5	0.5099 (0.0055)	0.5123 (0.0056)	0.4920 (0.0051)	0.5174 (0.0058)
	2.5	1.0	1.0	1.0	2.5494 (0.1382)	2.4718 (0.1156)	2.3749 (0.1216)	(0.1171)

Sample size (n)	Parameter (True value)				Methods of estimation			
	α	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
75	1.0	1.0	1.0	1.0	1.0145 (0.0143)	1.0141 (0.0139)	0.9874 (0.0132)	1.0207 (0.0143)
	3.0	0.5	0.5	0.5	3.0434 (0.1291)	3.0020 (0.1187)	2.9224 (0.1185)	3.0218 (0.1208)
	0.5	2.5	0.5	0.5	0.5072 (0.0036)	0.5089 (0.0036)	0.4954 (0.0034)	0.5122 (0.0037)
	2.5	1.0	1.0	1.0	2.5362 (0.0897)	2.4848 (0.0794)	2.4194 (0.0815)	2.5011 (0.0802)
100	1.0	1.0	1.0	1.0	1.0111 (0.0107)	1.0109 (0.0105)	0.9909 (0.0100)	1.0159 (0.0107)
	3.0	0.5	0.5	0.5	3.0334 (0.0965)	3.0026 (0.0906)	2.9428 (0.0902)	3.0175 (0.0918)
	0.5	2.5	0.5	0.5	0.5056 (0.0027)	0.5068 (0.0027)	0.4967 (0.0026)	0.5093 (0.0028)
	2.5	1.0	1.0	1.0	2.5278 (0.0670)	2.4895 (0.0611)	2.4402 (0.0622)	2.5018 (0.0616)
125	1.0	1.0	1.0	1.0	1.0081 (0.0081)	1.008 (0.008)	0.9920 (0.0078)	1.0120 (0.0082)
	3.0	0.5	0.5	0.5	3.0243 (0.0733)	2.9998 (0.0698)	2.9520 (0.0699)	3.0118 (0.0705)
	0.5	2.5	0.5	0.5	0.5041 (0.0020)	0.505 (0.002)	0.497 (0.002)	0.5070 (0.0021)
	2.5	1.0	1.0	1.0	2.5203 (0.0509)	2.4898 (0.0474)	2.4503 (0.0483)	2.4997 (0.0477)
150	1.0	1.0	1.0	1.0	1.007 (0.0070)	1.0070 (0.0069)	0.9937 (0.0067)	1.0103 (0.0070)
	3.0	0.5	0.5	0.5	3.0213 (0.0627)	3.0010 (0.0601)	2.9611 (0.0601)	3.0109 (0.0606)
	0.5	2.5	0.5	0.5	0.5036 (0.0017)	0.5044 (0.0017)	0.4977 (0.0017)	0.5061 (0.0018)
	2.5	1.0	1.0	1.0	2.5178 (0.0435)	2.4925 (0.0410)	2.4594 (0.0415)	2.5007 (0.0412)

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function

Looking at the results from table 1-3, one can see that the estimators of the shape parameter using QLF under Gamma, uniform and Jeffrey priors is better than the other estimators based on the fact that it has the lowest MSE despite the changes in the samples and chosen parameter values. This consistency in the result for Bayesian estimators (using QLF under Uniform, Jeffrey and gamma priors) is a proof that the approach is the more efficient for estimating the shape parameter compared to MLE and Bayesian with the other two loss functions. Also, based on the prior distributions it is found that the QLF under the gamma prior has the smallest MSEs compared to uniform and Jeffrey priors irrespective of the parameter values and the sample sizes and this excellent performance of the QLF is found to be consistent despite all differences.

Generally and conclusively, the results in Tables 1, 2 and 3 have proven that the average estimates of the shape parameter get closer to the true parameter value when sample size increases and the mean square errors (MSEs) all decrease as sample size increases which satisfies the first-order asymptotic theory. Similarly, Bayesian estimators and maximum likelihood estimators (MLEs) all become better when the sample size increases. In fact, for very large sample sizes the performances of these estimators are observed to be relatively the same for both methods of estimation.

5 Conclusion

This paper has derived Bayesian estimators for the shape parameter of exponential inverse exponential distribution by assuming Uniform, Jeffrey and gamma prior distributions with three loss functions which include Squared Error Loss Function, Quadratic Loss Function and Precautionary Loss Function. Posterior distributions and Bayes estimators of this parameter are derived using the priors and loss functions respectively. The efficiency of these estimators have been evaluated in relation to their mean square errors using the inverse transformation method of Monte Carlo Simulations with various parameter values and sample sizes. The results of the simulation and comparison show that using quadratic loss function gives estimators with the lowest MSEs under all the prior distributions (gamma, Jeffreys and uniform). Precisely, it is found that Bayesian Method using Quadratic Loss Function under gamma prior produces the best estimators of the shape parameter compared to estimators of Maximum Likelihood method, Squared Error Loss Function and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the chosen parameters values and the allocated sample sizes. This research also found that the variation in the values of the scale parameter of the distribution does not affect or change the performance of the estimators of the estimated shape parameter, however, it is recommended that since this study considers only the shape parameter of the exponential inverse exponential distribution, subsequent works should consider the scale parameter of the distribution due to the fact that in statistical applications of this model it will be very important to identify and understand the best method for estimating both the scale and shape parameters of the model.

Competing Interests

Authors have declared that no competing interests exist.

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