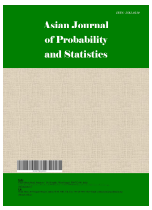

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DC Pension Plan with Refund of Contributions under Affine Interest Rate Model

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Authors' contributions

This work was carried out by the three authors. Authors UOI and OCM developed and solved the model. Author EEA did the theoretical analyses and interpretations and authors UOI and OCM wrote the literature of the work. All the authors approved the final manuscript.

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Abstract

This paper studies the optimal investment plan for a pension scheme with refund of contributions, stochastic salary and affine interest rate model. A modified model which allows for refund of contributions to death members' families is considered. In this model, the fund managers invest in a risk free (treasury) and two risky assets (stock and zero coupon bond) such that the price of the risky assets are modelled by geometric Brownian motions and the risk free interest rate is of affine structure. Using the game theoretic approach, an extended Hamilton Jacobi Bellman (HJB) equation which is a system of non linear PDE is established. Furthermore, the extended HJB equation is then solved by change of variable and variable separation technique to obtain explicit solutions of the optimal investment plan for the three assets using mean variance utility function. Finally, theoretical analyses of the impact of some sensitive parameters on the optimal investment plan are presented.

Keywords: Pension scheme; extended HJB equation; investment plan; refund clause; stochastic salary; affine interest rate.

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1 Introduction

The practice of defined contribution (DC) pension scheme has increase rapidly in so many countries across the world due to the fact that it provides a comfortable platform for its members to be involved in the day to day activities of the pension fund. The most enticing feature of the DC plan is that members are fully involved in planning for their retirement benefits. Since members benefits depend on the various investment in the financial market which may include cash, bond, loan and stock etc., there is need to develop and understand how best their funds can be invested for optimal profit with minimal risk. This has led to the study of optimal invest plan for a DC pension scheme.

In a DC scheme, the study of optimal investment plan with refund clauses have been explored by some authors under different assumptions which include [1], considered investment in one risk free asset and one risky asset and assumed the risk-free asset is modelled by geometric Brownian motion. [2], assumed the stock market price was modelled by Heston volatility and considered investment in both accumulation and distribution phase. [3], studied optimization problem with return of premium in a DC pension with multiple contributors; in their work, the stock market price was driven by constant elasticity of variance model (CEV) model. [4] solved the optimal investment problem for a DC pension plan with default risk and return of premiums clauses; they assumed they stock market price followed constant elasticity of variance (CEV) model. [5], investigated investment plan with return of premium clauses under inflation risk and volatility risk; they considered investment in a risk-free asset, the inflation index bond and the stock whose price was modelled by Heston volatility. [6], studied optimal investment plan for four different assets modelled by geometric Brownian motion whose return of contributions was with risk free interest. [7], studied investment strategies when the returned contributions are with predetermined interest; they assumed that the return contributions are with risk free interest. In this paper, we study the optimal investment plan with refund of contributions with stochastic salary and stochastic interest rate which is of affine structure.

The optimal investment plan in a DC pension plan with stochastic interest rate have been studied by some authors such as [8,9], where they assume the interest rate to be of Vasicek model. [10,11], studied the same problem but assume the interest rate to have an affine structure i.e a model with Vasicek and Cox – Ingeroll – Ross (CIR) model. [8], studied optimal investment plan with risk generated from the salary and inflation using vasicek model. [12], studied optimal investment plan with stochastic salary under affine interest rate. The effect of extra contribution on stochastic optimal investment strategies in a DC pension with stochastic salary under affine interest rate model was studied by [13], they considered a case where the extra contributions rate was with both stochastic and constant.

In this paper, we study the optimal investment plan with refund of contributions with stochastic salary and stochastic interest rate which is of affine structure. We will give theoretical analyses of the impact of some sensitive parameters on the optimal investment plan.

2 The Investment Plan Model

Starting with a complete and frictionless financial market that is continuously open over the fixed time interval $[0, T]$, for $T > 0$ representing the retirement time of a given shareholder.

We assume that the market is made up of risk free asset (cash) a zero coupon bond and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{B_r(t), B_s(t): t \geq 0\}$ is a standard two dimensional motion such that they orthogonal to each other. F is the filtration and denotes the information generated by the Brownian motion $\{B_r(t), B_s(t)\}$.

Let $\mathcal{C}(t)$ denote the price of the risk free asset at time t and it is modelled as follows

$$d\mathcal{C}(t) = r(t)\mathcal{C}(t)dt \qquad \mathcal{C}(0) > 0 \tag{2.1}$$

$r(t)$ is the short interest rate process and is given as

$$dr\ (t)\ =\ (a - br\ (t))\ dt - \sigma_r dB_r(t)\ , \tag{2.2}$$

$$\sigma_r = \sqrt{c_1 r\ (t)\ +\ c_2}\ ,\ t \geq 0,$$

Where a, b, (0), c_1, and $c_2$are positive real numbers. Here the term structure of the short interest rates is affine, which has been studied by [10,11,12,14].

Let $S(t)$ denote the price of stock and its dynamics is given based on the stochastic differential equation at $t \geq 0$ and the price process is described as follows:

$$dS(t)\ = S(t)[r(t)dt + \ \sigma_s(dB_s(t) + \lambda_1 dt) + n_1\sigma_r(dB_r(t) + \lambda_2\sigma_r dt)],\ S(0) = \ S_0 \tag{2.3}$$

with λ_1, λ_2, σ_s, n_1 are positive constant [10-12].

Next we consider a zero-coupon bond with maturity$T$, whose price at time $t$is denoted by $B(t,T), t \geq 0$, and it dynamics is given by the SDE below [10,11].

$$dB(t,T)\ = B(t,T)[r(t)dt + \ \sigma_B\big(T - t, r(t)\big)(dB_r(t) + \lambda_2\sigma_r dt)],\ B(T,T) = \ 1 \tag{2.4}$$

Where $\sigma_B(T - t, r(t)) = f(T - t)\sigma_r$ and

$$g(t) = \frac{2(e^{dt} - 1)}{d - (b - c_1\lambda_2) + e^{dt}(d + b - c_1\lambda_2)},\ d = \sqrt{((b - c_1\lambda_2)^2 + 2c_1} \tag{2.5}$$

Stochastic Salary: From the works of [8,9,11,12], we denote the salary at time $t$by $L(t)$ which is described by

$$dL(t)\ = L(t)[\delta_L(t, r\ (t))\ dt + \ n_2\sigma_r dB_r(t) + n_3\sigma_s dB_s(t)],\ L(0) = \ L_0 \tag{2.6}$$

where n_2,n_3 are real constants, representing instantaneous volatility which measures the risk sources of interest rate and stock affecting the salary. That is to say, the salary volatility is supposed to a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. This is in accordance with the assumption in [11] but is differs from those of [8,9] who also suggest that the salary was influence by non hedgeable risk source (i.e., non-financial market).

Also [12,13], assume that the instantaneous mean of the salary is such that $\delta_L\big(t, r(t)\big) = r(t) + \varpi_L$ where $\varpi_L$is a real constant.

Let $\mathcal{Z}(t)$ represent the accumulated wealth of the pension fund at time t andconsidering the time interval $[t, t + i]$, the differential form associated with the fund size is given as:

$$\mathcal{Z}(t + i) = \frac{1}{1 - i\mathfrak{M}_{\vartheta_0 + t}}\left(\begin{array}{c} imL(t) - tmL(t)i\mathfrak{M}_{\vartheta_0 + t} \\ +\mathcal{Z}(t)\left(\mu_1\frac{\mathcal{C}(t+i)}{\mathcal{C}(\mathrm{t})} + \mu_2\frac{\mathcal{S}(t+i)}{\mathcal{S}(t)} + \mu_3\frac{B(t+i)}{B(t)}\right) \end{array}\right) \tag{2.7}$$

$$\mathcal{Z}(t + i) = \left(\frac{i\mathfrak{M}_{\vartheta_0 + t}}{1 - i\mathfrak{M}_{\vartheta_0 + t}} + 1\right)\left(\begin{array}{c} imL(t) - tmL(t)i\mathfrak{M}_{\vartheta_0 + t} \\ +\mathcal{Z}(t)\left(\begin{array}{c} 1 + (1 - \mu_2 - \mu_3)\left(\frac{\mathcal{C}(t+i) - \mathcal{C}(\mathrm{t})}{\mathcal{C}(\mathrm{t})}\right) \\ +\mu_2\left(\frac{\mathcal{S}(t+i) - \mathcal{S}(t)}{\mathcal{S}(t)}\right) + \mu_3\left(\frac{B(t+i) - B(t)}{B(t)}\right) \end{array}\right) \end{array}\right) \tag{2.8}$$

Where μ_1,μ_2, and $\mu_3$are the optimal control planscash, equity and loan respectively such that $\mu_1 = 1 - \mu_2 - \mu_3$, m is the members' contributions received by the pension fund at any given time, ϑ_0, the initial age of

accumulation phase, T, the time frame of the accumulation period such that $\vartheta_0 + T$ is the end age. Suppose $i\mathfrak{M}_{\vartheta_0+t}$ is the mortality rate from time t to $t+i$, $tL(t)$ is the accumulated contributions at time t, $tmL(t)i\mathfrak{M}_{\vartheta_0+t}$ is the returned accumulated contributions of death members.

Following [1,2], we have

The conditional death probability ${}_tb_y = 1 - {}_ta_y = 1 - e^{-\int_0^t \pi(\vartheta_0+t+s)ds}$, where $\pi(t)$ is the force function of the mortality at time t, and for $i \to 0$,

$$i\mathfrak{M}_{\vartheta_0+t} = 1 - \exp\{-\int_0^i \pi(\vartheta_0 + t + s)ds\} \approx \pi(\vartheta_0 + t)i + O(i)$$

$$\frac{i\mathfrak{M}_{\vartheta_0+t}}{1 - i\mathfrak{M}_{\vartheta_0+t}} = \frac{1 - \exp\{-\int_0^i \pi(\vartheta_0 + t + s)ds\}}{\exp\{-\int_0^i \pi(\vartheta_0 + t + s)ds\}} = \exp\{\int_0^i \pi(\vartheta_0 + t + s)ds\} - 1 \approx \pi(\vartheta_0 + t)i + O(i)$$

$$i \to 0, \frac{i\mathfrak{M}_{\vartheta_0+t}}{1-i\mathfrak{M}_{\vartheta_0+t}} = \pi(\vartheta_0 + t)dt, i\mathfrak{M}_{\vartheta_0+t} = \pi(\vartheta_0 + t)dtmi \to mdt, \frac{\mathcal{C}(t+i)-\mathcal{C}(t)}{\mathcal{C}(t)} \to \frac{d\mathcal{C}(t)}{\mathcal{C}(t)},$$

$$\left(\frac{\mathcal{S}(t+i)-\mathcal{S}(t)}{\mathcal{S}(t)}\right) \to \frac{d\mathcal{S}(t)}{\mathcal{S}(t)}, \left(\frac{B(t+i)-B(t)}{B(t)}\right) \to \frac{dB(t)}{B(t)} \tag{2.9}$$

Substituting (2.9) into (2.8), we have

$$\mathcal{Z}(t+i) = (1 + \pi(\vartheta_0 + t)dt)\begin{pmatrix} mL(t)dt - tmL(t)\pi(\vartheta_0 + t)dt \\ +\mathcal{Z}(t)\begin{pmatrix} 1 + (1 - \mu_2 - \mu_3)\frac{d\mathcal{C}(t)}{\mathcal{C}(t)} \\ +\mu_2\frac{d\mathcal{S}(t)}{\mathcal{S}(t)} + \mu_3\frac{dB(t)}{B(t)} \end{pmatrix} \end{pmatrix} \tag{2.10}$$

Substituting (2.1), (2.3) and (2.4) into (2.10), we have

$$d\mathcal{Z}(t) = \begin{pmatrix} \left\{\mathcal{Z}(t)\begin{pmatrix} r(t) + \mu_2(\lambda_1\sigma_s + \lambda_2{\sigma_r}^2 n_1) \\ +\mu_3\lambda_2\sigma_r\sigma_B \\ +\frac{1}{\vartheta-\vartheta_0-t} \end{pmatrix}\right\}dt + \\ +mL(t)\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) \\ \mathcal{Z}(t)\left((\sigma_B\mu_3 + \mu_2\sigma_r n_1)dB_r(t) + \sigma_s\mu_2 dB_s(t)\right) \end{pmatrix} \mathcal{Z}(0) = z_0 \tag{2.11}$$

Where ϑ is the maximal age of the life table and $\pi(t)$is the force function given by

$$\pi(t) = \frac{1}{\vartheta - t}, \qquad 0 \le t < \vartheta \tag{2.12}$$

When it is time for retirement the contributor will be interested in preserving his standard of living and will be interested in his retirement income relative to his predetermine salary. Assume we consider the contributor's salary as a numeraire. Let the relative wealth be defined as follows

$$X(t) = \frac{\mathcal{Z}(t)}{L(t)} \tag{2.13}$$

Applying product rule and Ito's formula to (2.13) and making use of (2.6) and (2.11) we arrive at the following equation

$$dX(t) = \begin{pmatrix} \left\{ X(t)\left(\theta_1 + \mu_2\theta_3 + \mu_3\theta_2 + \frac{1}{\vartheta - \vartheta_0 - t}\right) \right\} dt + \\ +mL(t)\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) \\ X(t)\big((\sigma_B\mu_3 + \mu_2\sigma_r \mathrm{n}_1 - \sigma_r \mathrm{n}_2)dB_r(t) + \sigma_s(\mu_2 - \mathrm{n}_3)dB_s(t)\big) \end{pmatrix}, X(0) = \frac{Z(0)}{L(0)} = \frac{z_0}{l_0} \quad (2.14)$$

$$\theta_1 = r(t) - \delta_L + {n_2}^2{\sigma_r}^2 + {n_3}^2{\sigma_S}^2$$

$$\theta_2 = \sigma_B\sigma_r(\lambda_2 - n_2) \quad (2.15)$$

$$\theta_3 = \lambda_1\sigma_s + \lambda_2{\sigma_r}^2 n_1 + {\sigma_S}^2 n_3 - n_1 n_2 {\sigma_r}^2$$

3 Optimization Problem

Consider a pension fund manager whose interest is to maximize his profit while penalising risk by using the mean-variance utility function given as

$$\mathcal{F}(T,x,r) = \sup_{\mu}\{E_{t,x,r}X^{\mu}(T) - Var_{t,x,r}X^{\mu}(T)\} \quad (3.1)$$

Applying the game theoretic method described in [15], the mean-variance control problem in (3.1) is similar to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{F}(T,x,r)$.

$$\begin{cases} \mathcal{G}(t,x,r,\mu) = E_{t,x,r}[X^{\mu}(T)] - \dfrac{\beta}{2}Var_{t,x,r}[X^{\mu}(T)] \\ \mathcal{G}(t,x,r,\mu) = E_{t,x,r}[X^{\mu}(T)] - \dfrac{\beta}{2}\left(E_{t,x,r}[X^{\mu}(T)^2] - (E_{t,x,r}[X^{\mu}(T)])^2\right) \\ \mathcal{F}(T,x,r) = \sup\limits_{\mu}\mathcal{G}(t,x,r,\mu) \end{cases}$$

From [15], the optimal control plan μ^* satisfies:

$$\mathcal{F}(T,x,r) = \sup_{\mu}\mathcal{G}(t,x,r,\mu^*)$$

β is a constant representing risk aversion coefficient of the members.

Let $p^{\mu}(t,x,r) = E_{t,x,r}[X^{\mu}(T)]$, $q^{\mu}(t,x,r) = E_{t,x,r}[X^{\mu}(T)^2]$ then

$$\mathcal{F}(T,x,r) = \sup_{\mu} u\big(t,x,r,p^{\mu}(t,x,r),q^{\mu}(t,x,r)\big)$$

Where,

$$u(t,x,r,p,q) = p - \frac{\beta}{2}(q - p^2) \quad (3.2)$$

And

$$u_t = u_r = u_{rr} = u_{rp} = u_{rq} = u_{pq} = u_{qq} = 0, u_p = 1 + \beta p, u_{pp} = \beta, u_q = -\frac{\beta}{2} \quad (3.3)$$

Theorem 3.1 (verification theorem). Suppose, there exist three real functions $\mathcal{U}, \mathcal{V}, \mathcal{W}: [0,T] \times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\sup_{\mu}\begin{Bmatrix}(\mathcal{U}_t-u_t)+\left[x\left(\theta_1+\mu_2\theta_3+\mu_3\theta_2+\frac{1}{\vartheta-\vartheta_0-t}\right)+m\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right](\mathcal{U}_x-u_x)\\ +(a-br)(\mathcal{U}_r-u_r)+\frac{1}{2}\sigma_r^2(\mathcal{U}_{rr}-\mathcal{A}_{rr})\\ +\frac{1}{2}x^2[(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)^2+\sigma_s^2(\mu_2-\mathrm{n}_3)^2](\mathcal{U}_{xx}-\mathcal{A}_{xx})\\ -x\sigma_r(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)(\mathcal{U}_{rx}-\mathcal{A}_{rx})\\ \mathcal{U}(T,x,r)=u(t,x,r,x,x^2)\end{Bmatrix}=0 \tag{3.4}$$

Where,

$$\mathcal{A}_{xx}=u_{xx}+2u_{xp}p_x+2u_{xq}q_x+u_{pp}p_x^2+2u_{pq}p_xq_x+u_{qq}q_x^2=\gamma\mathcal{V}_x^2 \tag{3.5}$$

$$\mathcal{A}_{xr}=u_{xr}+u_{xp}p_r+u_{xq}q_r+u_{pr}p_x+u_{qr}q_x+u_{pp}p_xp_r+u_{pq}p_xq_r \\ +u_{pq}p_rq_x+u_{qq}q_rq_x=\gamma\mathcal{V}_x\mathcal{V}_r \tag{3.6}$$

$$\mathcal{A}_{rr}=u_{rr}+2u_{rp}p_r+2u_{rq}q_r+u_{pp}p_r^2+2u_{pq}p_rq_r+u_{qq}q_r^2=\gamma\mathcal{V}_r^2 \tag{3.7}$$

$$\begin{Bmatrix}\mathcal{V}_t+\left[x\left(\theta_1+\mu_2\theta_3+\mu_3\theta_2+\frac{1}{\vartheta-\vartheta_0-t}\right)+m\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right]\mathcal{V}_x\\ +(a-br)\mathcal{V}_r+\frac{1}{2}\sigma_r^2\mathcal{V}_{rr}\\ +\frac{1}{2}x^2[(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)^2+\sigma_s^2(\mu_2-\mathrm{n}_3)^2]\mathcal{V}_{xx}\\ -x\sigma_r(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)\mathcal{V}_{rx}\\ \mathcal{V}(T,x,r)=x\end{Bmatrix}=0 \tag{3.8}$$

$$\begin{Bmatrix}\mathcal{W}_t+\left[x\left(\theta_1+\mu_2\theta_3+\mu_3\theta_2+\frac{1}{\vartheta-\vartheta_0-t}\right)+m\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right]\mathcal{W}_x\\ +(a-br)\mathcal{W}_r+\frac{1}{2}\sigma_r^2\mathcal{W}_{rr}\\ +\frac{1}{2}x^2[(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)^2+\sigma_s^2(\mu_2-\mathrm{n}_3)^2]\mathcal{W}_{xx}\\ -x\sigma_r(\sigma_B\mu_3+\mu_2\sigma_r\mathrm{n}_1-\sigma_r\mathrm{n}_2)\mathcal{W}_{rx}\\ \mathcal{W}(T,x,r)=x^2\end{Bmatrix}=0 \tag{3.9}$$

Then $\mathcal{F}(T,x,r)=\mathcal{U}(T,x,r), p^{\mu^*}=\mathcal{V}(T,x,r), q^{\mu^*}=\mathcal{W}(T,x,r)$for the optimal investment plan μ^*.

The details of the proof can be found in [16-18].

Substituting (3.3) into (3.4) and differentiating it with respect to μ_2 and μ_3 and solving for $\mu_2$and μ_3, we have

$$\mu_2{}^*=\frac{x\sigma_s n_3(\mathcal{U}_{xx}-\mathcal{A}_{xx})-(\lambda_1+n_3\sigma_s^2)\mathcal{U}_x}{(\mathcal{U}_{xx}-\mathcal{A}_{xx})x\sigma_s} \tag{3.10}$$

$$\mu_3{}^*=\frac{x\sigma_r(n_2-n_1n_3)(\mathcal{U}_{xx}-\mathcal{A}_{xx})+\theta_4\sigma_r\mathcal{U}_x+\sigma_r(\mathcal{U}_{xr}-\mathcal{A}_{xr})}{(\mathcal{U}_{xx}-\mathcal{A}_{xx})x\sigma_B} \tag{3.11}$$

$$\theta_4=\frac{\sigma_s n_2+n_1\lambda_1+n_1n_3\sigma_s^2-\lambda_2\sigma_s}{\sigma_s} \tag{3.12}$$

Substituting (3.4), (3.8) and (3.9) into (3.3), we have

$$\mathcal{U}_\mathrm{t}+\mathcal{U}_x\left[x(\varphi_0+\frac{1}{\vartheta-\vartheta_0-t})+m\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right]+\frac{\mathcal{U}_x^2}{2(\mathcal{U}_{xx}-\gamma\mathcal{V}_x^2)}(2\varphi_1-(c_1r\,(t)\;+\;c_2)(\lambda_2-\mathrm{n}_2)^2)$$

$$+(\lambda_2 - \mathrm{n}_2)(c_1 r\,(t)\; +\; c_2)\frac{\mathcal{U}_x(\mathcal{U}_{rx}-\gamma\mathcal{V}_x\mathcal{V}_r)}{2(\mathcal{U}_{xx}-\gamma\mathcal{V}_x^2)} - (c_1 r\,(t)\; +\; c_2)\frac{(\mathcal{U}_{rx}-\gamma\mathcal{V}_x\mathcal{V}_r)^2}{2(\mathcal{U}_{xx}-\gamma\mathcal{V}_x^2)} + (a-br)\mathcal{U}_r + \frac{1}{2}(c_1 r\,(t)\; +\; c_2)(\mathcal{U}_{rr}-\gamma\mathcal{V}_r^2) = 0 \tag{3.13}$$

$$\mathcal{V}_{\mathrm{t}} + \mathcal{V}_x\left[x(\varphi_0 + \frac{1}{\vartheta-\vartheta_0-t}) + m\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] - \frac{\mathcal{U}_x\mathcal{V}_x}{(\mathcal{U}_{xx}-\gamma\mathcal{V}_x^2)}(2\varphi_1 - (c_1 r\,(t)\; +\; c_2)(\lambda_2-\mathrm{n}_2)^2)$$
$$+(\lambda_2 - \mathrm{n}_2)(c_1 r\,(t)\; +\; c_2)\frac{\mathcal{V}_x(\mathcal{U}_{rx}-\gamma\mathcal{V}_x\mathcal{V}_r)}{2(\mathcal{U}_{xx}-\gamma\mathcal{V}_x^2)} + (a-br)\mathcal{V}_r + \frac{1}{2}(c_1 r\,(t)\; +\; c_2)\mathcal{V}_{rr} + \frac{1}{2}x^2\mathcal{V}_{xx}\big[(\sigma_B\mu_3 +$$
$$\mu 2(c1r\,(t) + c2)\mathrm{n}1 - (c1r\,(t) + c2)\mathrm{n}2)2 + \sigma\mathrm{s}2\mu 2 - \mathrm{n}32 - x(c1r\,(t) + c2)\mathcal{V}xr(\sigma B\mu 3 + \mu 2\mathrm{n}1(c1r\,(t) + c2) - \mathrm{n}2(c1r\,(t) + c2)) = 0 \tag{3.14}$$

$$\varphi_0 = n_3\lambda_1\sigma_s + \lambda_2\sigma_r^2 n_2 + 2n_3^2\sigma_s^2 - \varpi_L$$

$$\varphi_1 = \frac{1}{2}\sigma_s^4 n_3^2 - n_3^2\sigma_s^3 - n_3\lambda_1\sigma_s - \frac{1}{2}{\lambda_1}^2$$

$$\sigma_r^2 = c_1 r\,(t)\; +\; c_2$$

Next, we conjecture a solution for $\mathcal{U}(t,x,r)$ and $\mathcal{V}(t,x,r)$ as follows:

$$\begin{cases} \mathcal{U}(t,x,r) = x\mathcal{H}(t) + \frac{r}{\beta}\mathcal{M}(t) + \frac{1}{\beta}\mathcal{N}(t)\mathcal{H}(T) = 1,\ \mathcal{M}(T) = 0,\ \mathcal{N}(t) = 0 \\ \mathcal{V}(t,x,r) = x\overline{\mathcal{H}}(t) + \frac{r}{\beta}\overline{\mathcal{M}}(t) + \frac{1}{\beta}\overline{\mathcal{N}}(t)\overline{\mathcal{H}}(T) = 1,\ \overline{\mathcal{M}}(T) = 0,\ \overline{\mathcal{N}}(t) = 0 \\ \mathcal{U}_{\mathrm{t}} = x\mathcal{H}_{\mathrm{t}} + \frac{r}{\beta}\mathcal{M}_{\mathrm{t}} + \frac{1}{\beta}\mathcal{N}_{\mathrm{t}}\mathcal{U}_x = \mathcal{H}(t), \mathcal{U}_r = \frac{1}{\beta}\mathcal{M}(t), \mathcal{U}_{xx} = \mathcal{U}_{xr} = \mathcal{U}_{rr} = 0 \\ \mathcal{V}_{\mathrm{t}} = x\overline{\mathcal{H}}_{\mathrm{t}} + \frac{r}{\beta}\overline{\mathcal{M}_t} + \frac{1}{\beta}\overline{\mathcal{N}}_t\mathcal{U}_x = \overline{\mathcal{H}}(t), \mathcal{V}_r = \frac{1}{\beta}\overline{\mathcal{M}}(t), \mathcal{V}_{xx} = \mathcal{V}_{xr} = \mathcal{V}_{rr} = 0 \end{cases} \tag{3.15}$$

Substituting (3.15) into (3.13) and (3.14), we have

$$\begin{cases} x[\mathcal{H}_t(t) + \left(\varphi_0 + \frac{1}{\vartheta-\vartheta_0-t}\right)\mathcal{H}(t)] = 0 \\ \frac{r}{\beta}[\mathcal{M}_{\mathrm{t}} - b\mathcal{M}(t) + \frac{\mathcal{H}^2(t)}{2\overline{\mathcal{H}}^2(t)}c_1(\lambda_2-\mathrm{n}_2)^2 + c_1(\lambda_2-\mathrm{n}_2)\overline{\mathcal{M}}(t)] = 0 \\ \frac{1}{\beta}[\mathcal{N}_{\mathrm{t}} + m\beta\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\mathcal{H}(t) - \varphi_1\frac{\overline{\mathcal{H}}(t)\mathcal{H}(t)}{\overline{\mathcal{H}}^2(t)} + \frac{\mathcal{H}^2(t)}{2\overline{\mathcal{H}}^2(t)}c_2(\lambda_2-\mathrm{n}_2)^2 \\ \qquad +a\mathcal{M}(t) + c_2(\lambda_2-\mathrm{n}_2)\overline{\mathcal{M}}(t)] = 0 \end{cases} \tag{3.16}$$

$$\begin{cases} x[\overline{\mathcal{H}}(t) + \left(\varphi_0 + \frac{1}{\vartheta-\vartheta_0-t}\right)\overline{\mathcal{H}}(t)] = 0 \\ \frac{r}{\beta}[\overline{\mathcal{M}}(t) - b\overline{\mathcal{M}}(t) + \frac{\overline{\mathcal{H}}(t)\mathcal{H}(t)}{\overline{\mathcal{H}}^2(t)}c_1(\lambda_2-\mathrm{n}_2)^2 + c_1(\lambda_2-\mathrm{n}_2)\overline{\mathcal{M}}(t)] = 0 \\ \frac{1}{\beta}[\overline{\mathcal{N}}(t) + m\beta\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\overline{\mathcal{H}}(t) - 2\varphi_1\frac{\overline{\mathcal{H}}(t)\mathcal{H}(t)}{\overline{\mathcal{H}}^2(t)} + a\overline{\mathcal{M}}(t) + c_2(\lambda_2-\mathrm{n}_2)\overline{\mathcal{M}}(t)] = 0 \end{cases} \tag{3.17}$$

Solving (3.16) and (3.17), we have

$$\mathcal{H}(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)} \tag{3.18}$$

$$\mathcal{M}(t) = \frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)} - \frac{1}{b}\right] + \frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b} + \begin{bmatrix}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b} \\ -\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b}\end{bmatrix}e^{b(t-T)} \tag{3.19}$$

$$\mathcal{N}(t)=\begin{pmatrix}\left[\varphi_1-\frac{c_2(\lambda_2-\mathrm{n}_2)^2}{2}+\frac{c_1c_2(\lambda_2-\mathrm{n}_2)^3}{c_1(\lambda_2-\mathrm{n}_2)-b}+\frac{ac_1(\lambda_2-\mathrm{n}_2)^2}{b(c_1(\lambda_2-\mathrm{n}_2)-b)}-\frac{ac_1(\lambda_2-\mathrm{n}_2)^2}{2b}\right](T-t)\\ +\frac{c_1c_2(\lambda_2-\mathrm{n}_2)^3+ac_1(\lambda_2-\mathrm{n}_2)^2}{(c_1(\lambda_2-\mathrm{n}_2)-b)^2}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right]\\ -\frac{a}{b}\begin{bmatrix}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\\ -\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b}\end{bmatrix}(e^{b(t-T)}-1)\\ +\frac{m\beta}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\left[(\vartheta-\vartheta_0-2(t+1))e^{\varphi_0(T-t)}-\left(\vartheta-\vartheta_0-2(T+1)\right)\right]\end{pmatrix} \tag{3.20}$$

$$\overline{\mathcal{H}}(t)=\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)} \tag{3.21}$$

$$\overline{\mathcal{M}}(t)=\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right] \tag{3.22}$$

$$\overline{\mathcal{N}}(t)=\begin{pmatrix}\left[2\varphi_1+\frac{c_1(a+c_2(\lambda_2-\mathrm{n}_2))}{c_1(\lambda_2-\mathrm{n}_2)-b}\right](T-t)+\frac{c_1(a+c_2(\lambda_2-\mathrm{n}_2))}{(c_1(\lambda_2-\mathrm{n}_2)-b)^2}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right]\\ +\frac{m\beta}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\left[(\vartheta-\vartheta_0-2(t+1))e^{\varphi_0(T-t)}-\left(\vartheta-\vartheta_0-2(T+1)\right)\right]\end{pmatrix} \tag{3.23}$$

$$\mathcal{U}(t,x,r)=\begin{pmatrix}\frac{1}{\beta}\begin{pmatrix}r\begin{bmatrix}\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-\frac{1}{b}\right]+\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b}\\ +\begin{bmatrix}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\\ -\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b}\end{bmatrix}e^{b(t-T)}\end{bmatrix}\\ +\begin{pmatrix}\left[\varphi_1-\frac{c_2(\lambda_2-\mathrm{n}_2)^2}{2}+\frac{c_1c_2(\lambda_2-\mathrm{n}_2)^3}{c_1(\lambda_2-\mathrm{n}_2)-b}+\frac{ac_1(\lambda_2-\mathrm{n}_2)^2}{b(c_1(\lambda_2-\mathrm{n}_2)-b)}-\frac{ac_1(\lambda_2-\mathrm{n}_2)^2}{2b}\right](T-t)\\ +\frac{c_1c_2(\lambda_2-\mathrm{n}_2)^3+ac_1(\lambda_2-\mathrm{n}_2)^2}{(c_1(\lambda_2-\mathrm{n}_2)-b)^2}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right]\\ -\frac{a}{b}\begin{bmatrix}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\\ -\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{2b}\end{bmatrix}(e^{b(t-T)}-1)\end{pmatrix}\end{pmatrix}\\ \left[x\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)}+\frac{m}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\begin{bmatrix}\left(\vartheta-\vartheta_0-2(t+1)\right)e^{\varphi_0(T-t)}\\ -\left(\vartheta-\vartheta_0-2(T+1)\right)\end{bmatrix}\right]\end{pmatrix} \tag{3.24}$$

$$\mathcal{V}(t,x,r)=\begin{pmatrix}\frac{1}{\beta}\begin{pmatrix}r\left[\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right]\right]\\ +\begin{pmatrix}\left[2\varphi_1+\frac{c_1(a+c_2(\lambda_2-\mathrm{n}_2))}{c_1(\lambda_2-\mathrm{n}_2)-b}\right](T-t)\\ +\frac{c_1(a+c_2(\lambda_2-\mathrm{n}_2))}{(c_1(\lambda_2-\mathrm{n}_2)-b)^2}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right]\end{pmatrix}\end{pmatrix}\\ \left[x\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)}+\frac{m}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\begin{bmatrix}\left(\vartheta-\vartheta_0-2(t+1)\right)e^{\varphi_0(T-t)}\\ -\left(\vartheta-\vartheta_0-2(T+1)\right)\end{bmatrix}\right]\end{pmatrix} \tag{3.25}$$

From (3.15), we have

$$\mathcal{U}_x=\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)},\mathcal{U}_{xx}=\mathcal{U}_{xr}=0,\mathcal{V}_x=\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)},\mathcal{V}_{rr}=0,$$
$$\mathcal{V}_r=\frac{1}{\beta}\frac{c_1(\lambda_2-\mathrm{n}_2)^2}{c_1(\lambda_2-\mathrm{n}_2)-b}\left[e^{(c_1(\lambda_2-\mathrm{n}_2)-b)(T-t)}-1\right] \tag{3.26}$$

Substituting (3.26) into (3.10) and (3.11), we have

$$\mu_2^{\ *} = n_3 - \left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta x \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)} \tag{3.27}$$

$$\mu_3^{\ *} = \frac{\sigma_r(n_2 - n_1 n_3)}{\sigma_B} - \frac{\sigma_r\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)}\left(\theta_4 - \frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right]\right)}{\beta x \sigma_B} \tag{3.28}$$

lemma 3.1 The optimal Investment plans for the three assets are given as

$$\mu_1^{\ *} = 1 - \mu_2^{\ *} - \mu_3^{\ *}$$
$$\mu_2^{\ *} = n_3 - l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)}$$
$$\mu_3^{\ *} = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1 n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)}\right)\right][1 + h(t)]$$

Where

$$h(t) = \frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right]$$
$$\varphi_0 = n_3\lambda_1\sigma_s + \lambda_2 n_2(c_1 r \ + \ c_2) + 2n_3^2\sigma_s^2 + r - \mu_L$$
$$\sigma_r^2 = \ c_1 r \ + \ c_2$$

Lemma 3.2 The efficient frontier of the pension plan is given

$$E_{t,x,r}[X^{\mu^*}(T)] = \left(\begin{array}{c} \left(\begin{array}{c} r\left[\frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right]\right] \\ + \left(\begin{array}{c} \left[2\varphi_1 + \frac{c_1(a + c_2(\lambda_2 - \mathrm{n}_2))}{c_1(\lambda_2 - \mathrm{n}_2) - b}\right](T-t) \\ + \frac{c_1(a + c_2(\lambda_2 - \mathrm{n}_2))}{(c_1(\lambda_2 - \mathrm{n}_2) - b)^2}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right] \end{array}\right) \end{array}\right) \\ \times \sqrt{\frac{Var_{t,x,r}[X^{\mu^*}(T)]}{\left(\begin{array}{c} 2r\left[\begin{array}{c} \left(\frac{1}{b} - 1\right)\frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b} \\ - \frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{2b} \end{array}\right](1 - e^{b(t-T)}) \\ + 2\left[\varphi_1 + \frac{c_2(\lambda_2 - \mathrm{n}_2)^2}{2} + \frac{c_1(a + c_2(\lambda_2 - \mathrm{n}_2)) - c_1c_2(\lambda_2 - \mathrm{n}_2)^3}{c_1(\lambda_2 - \mathrm{n}_2) - b} - \frac{ac_1(\lambda_2 - \mathrm{n}_2)^2}{b(c_1(\lambda_2 - \mathrm{n}_2) - b)} + \frac{ac_1(\lambda_2 - \mathrm{n}_2)^2}{2b}\right](T-t) \\ + 2\frac{c_1(a + c_2(\lambda_2 - \mathrm{n}_2)) - c_1c_2(\lambda_2 - \mathrm{n}_2)^3 - ac_1(\lambda_2 - \mathrm{n}_2)^2}{(c_1(\lambda_2 - \mathrm{n}_2) - b)^2}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right] \\ + \frac{2a}{b}\left[\begin{array}{c} \left(\frac{1}{b} - 1\right)\frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b} \\ - \frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{2b} \end{array}\right](e^{b(t-T)} - 1) \end{array}\right)}} \\ \left[x\left(\frac{\vartheta - \vartheta_0 - t}{\vartheta - \vartheta_0 - T}\right) e^{\varphi_0(T-t)} + \frac{m}{\varphi_0}\left(\frac{1}{\vartheta - \vartheta_0 - T}\right)\left[\begin{array}{c} \left(\vartheta - \vartheta_0 - 2(t+1)\right)e^{\varphi_0(T-t)} \\ -\left(\vartheta - \vartheta_0 - 2(T+1)\right) \end{array}\right]\right] \end{array}\right) \tag{3.29}$$

Proof. Recall that

$$Var_{t,x,r}[X^{\mu^*}(T)] = E_{t,x,r}[X^{\mu^*}(T)^2] - (E_{t,x,r}[X^{\mu^*}(T)])^2$$
$$Var_{t,x,r}[X^{\mu^*}(T)] = \frac{2}{\gamma}[\mathcal{V}(t,x,r) - \mathcal{U}(t,x,r)] \tag{3.30}$$

Substituting (3.24) and (3.25) into (3.30), we have

$$Var_{t,x,r}[X^{\mu^*}(T)] = \frac{1}{\beta^2}\left(\begin{array}{c} 2r\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](1-e^{b(t-T)}) \\ +2\left[\varphi_1+\frac{c_2(\lambda_2-n_2)^2}{2}+\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3}{c_1(\lambda_2-n_2)-b}-\frac{ac_1(\lambda_2-n_2)^2}{b(c_1(\lambda_2-n_2)-b)}+\frac{ac_1(\lambda_2-n_2)^2}{2b}\right](T-t) \\ +2\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3-ac_1(\lambda_2-n_2)^2}{(c_1(\lambda_2-n_2)-b)^2}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right] \\ +\frac{2a}{b}\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](e^{b(t-T)}-1)\end{array}\right) \tag{3.31}$$

$$\frac{1}{\beta} = \sqrt{\frac{Var_{t,x,r}[X^{\mu^*}(T)]}{\left(\begin{array}{c} 2r\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](1-e^{b(t-T)}) \\ +2\left[\varphi_1+\frac{c_2(\lambda_2-n_2)^2}{2}+\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3}{c_1(\lambda_2-n_2)-b}-\frac{ac_1(\lambda_2-n_2)^2}{b(c_1(\lambda_2-n_2)-b)}+\frac{ac_1(\lambda_2-n_2)^2}{2b}\right](T-t) \\ +2\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3-ac_1(\lambda_2-n_2)^2}{(c_1(\lambda_2-n_2)-b)^2}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right] \\ +\frac{2a}{b}\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](e^{b(t-T)}-1)\end{array}\right)}} \tag{3.32}$$

$$E_{t,x,r}[X^{\mu^*}(T)] = \mathcal{V}(t,x,r) \tag{3.33}$$

Substituting (3.25) into (3.33), we have

$$E_{t,x,r}[X^{\mu^*}(T)] = \left(\begin{array}{c} \frac{1}{\beta}\left(\begin{array}{c} r\left[\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right]\right] \\ +\left(\begin{array}{c}\left[2\varphi_1+\frac{c_1(a+c_2(\lambda_2-n_2))}{c_1(\lambda_2-n_2)-b}\right](T-t) \\ +\frac{c_1(a+c_2(\lambda_2-n_2))}{(c_1(\lambda_2-n_2)-b)^2}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right]\end{array}\right)\end{array}\right) \\ \left[x\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)}+\frac{m}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\left[\begin{array}{c}\left(\vartheta-\vartheta_0-2(t+1)\right)e^{\varphi_0(T-t)} \\ -\left(\vartheta-\vartheta_0-2(T+1)\right)\end{array}\right]\right]\end{array}\right) \tag{3.34}$$

Substitute (3.32) in (3.34), we have:

$$E_{t,x,r}[X^{\mu^*}(T)] = \left(\begin{array}{c} \left(\begin{array}{c} r\left[\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right]\right] \\ +\left(\begin{array}{c}\left[2\varphi_1+\frac{c_1(a+c_2(\lambda_2-n_2))}{c_1(\lambda_2-n_2)-b}\right](T-t) \\ +\frac{c_1(a+c_2(\lambda_2-n_2))}{(c_1(\lambda_2-n_2)-b)^2}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right]\end{array}\right)\end{array}\right) \\ \times\sqrt{\frac{Var_{t,x,r}[X^{\mu^*}(T)]}{\left(\begin{array}{c} 2r\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](1-e^{b(t-T)}) \\ +2\left[\varphi_1+\frac{c_2(\lambda_2-n_2)^2}{2}+\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3}{c_1(\lambda_2-n_2)-b}-\frac{ac_1(\lambda_2-n_2)^2}{b(c_1(\lambda_2-n_2)-b)}+\frac{ac_1(\lambda_2-n_2)^2}{2b}\right](T-t) \\ +2\frac{c_1(a+c_2(\lambda_2-n_2))-c_1c_2(\lambda_2-n_2)^3-ac_1(\lambda_2-n_2)^2}{(c_1(\lambda_2-n_2)-b)^2}\left[e^{(c_1(\lambda_2-n_2)-b)(T-t)}-1\right] \\ +\frac{2a}{b}\left[\begin{array}{c}\left(\frac{1}{b}-1\right)\frac{c_1(\lambda_2-n_2)^2}{c_1(\lambda_2-n_2)-b} \\ -\frac{c_1(\lambda_2-n_2)^2}{2b}\end{array}\right](e^{b(t-T)}-1)\end{array}\right)}} \\ \left[x\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{\varphi_0(T-t)}+\frac{m}{\varphi_0}\left(\frac{1}{\vartheta-\vartheta_0-T}\right)\left[\begin{array}{c}\left(\vartheta-\vartheta_0-2(t+1)\right)e^{\varphi_0(T-t)} \\ -\left(\vartheta-\vartheta_0-2(T+1)\right)\end{array}\right]\right]\end{array}\right)$$

4 Theoretical Analysis

In this section, we present some lemmas to demonstrate the impact of some parameters on the optimal investment plan.

Lemma 4.1 Suppose $z > 0, \beta > 0, \varphi_0 > 0, t \in [0,T], \vartheta > 0, \vartheta_0 > 0, n_3 > 0, l > 0, \sigma_s > 0$, $\lambda_1 > 0$ and $\vartheta - \vartheta_0 - t > \vartheta - \vartheta_0 - T$ then

(i) $\frac{\partial \mu_2^*}{\partial z} < 0$ (ii) $\frac{\partial \mu_2^*}{\partial \beta} < 0$ (iii) $\frac{\partial \mu_2^*}{\partial l} > 0$ (iv) $\frac{\partial \mu_2^*}{\partial r} > 0$

Proof (i) Recall that $\mu_2^* = n_3 + l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$

$$\frac{\partial \mu_2^*}{\partial z} = -l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z^2 \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$$

Since $l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z^2 \sigma_s}\right) > 0$, $\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) > 0$ and $e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_2^*}{\partial z} = -l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z^2 \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)} < 0$$

Hence $\frac{\partial \mu_2^*}{\partial z} < 0$

(ii) Recall that $\mu_2^* = n_3 + l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$

$$\frac{\partial \mu_2^*}{\partial \beta} = -l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta^2 z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$$

Since $l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta^2 z \sigma_s}\right) > 0$, $\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) > 0$ and $e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_2^*}{\partial \beta} = -l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta^2 z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)} < 0$$

Hence $\frac{\partial \mu_2^*}{\partial \beta} < 0$

(iii) Recall that $\mu_2^* = n_3 + l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$

$$\frac{\partial \mu_2^*}{\partial l} = \left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}$$

Since $\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right) > 0$, $\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) > 0$ and $e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_2{}^*}{\partial l} = \left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)} > 0$$

Hence $\frac{\partial \mu_2{}^*}{\partial l} > 0$

(iv) $\mu_2{}^* = n_3 + l\left(\frac{\lambda_1+n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) e^{\varphi_0(t-T)}$

$$\frac{\partial \mu_2{}^*}{\partial r} = l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)(n_2 c_1 \lambda_1 + 1) e^{\varphi_0(t-T)}$$

Where $\varphi_0 = n_3\lambda_1\sigma_s + \lambda_2 n_2(c_1 r + c_2) + 2n_3^2\sigma_s^2 + r - \mu_L$

Since $l\left(\frac{\lambda_1+n_3\sigma_s^2}{\beta z^2 \sigma_s}\right) > 0$, $\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) > 0$ *and* $(n_2 c_1 \lambda_1 + 1) e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_2{}^*}{\partial r} = l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)(n_2 c_1 \lambda_1 + 1) e^{\varphi_0(t-T)} > 0$$

Hence $\frac{\partial \mu_2{}^*}{\partial r} > 0$

Lemma 4.2 Suppose $z > 0, \theta_4 > 0,\ \beta > 0, \varphi_0 > 0, t \in [0,T], \vartheta > 0, \vartheta_0 > 0, n_1 > 0, n_2 > 0, n_3 > 0, l > 0\ \sigma_s > 0, \lambda_1 > 0$ and $\vartheta - \vartheta_0 - t > \vartheta - \vartheta_0 - T$ then

(i) $\frac{\partial \mu_2{}^*}{\partial z} < 0$ (ii) $\frac{\partial \mu_2{}^*}{\partial \beta} < 0$ (iii) $\frac{\partial \mu_2{}^*}{\partial l} > 0$ (iv) $\frac{\partial \mu_2{}^*}{\partial r} > 0$

Proof Recall that

$$\mu_3{}^* = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1 n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)}\right)\right][1 + h(t)]$$

Where

$$h(t) = \frac{c_1(\lambda_2 - \mathrm{n}_2)^2}{c_1(\lambda_2 - \mathrm{n}_2) - b}\left[e^{(c_1(\lambda_2 - \mathrm{n}_2) - b)(T-t)} - 1\right]$$

$$\varphi_0 = n_3\lambda_1\sigma_s + \lambda_2 n_2(c_1 r + c_2) + 2n_3^2\sigma_s^2 + r - \mu_L$$

$$\sigma_r^2 = c_1 r + c_2$$

(i) $\mu_3{}^* = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1 n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) e^{\varphi_0(t-T)}\right)\right][1 + h(t)]$

$$\frac{\partial \mu_3{}^*}{\partial z} = -\left[\frac{l\sigma_r\theta_4}{\sigma_B\beta z^2}\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right) e^{\varphi_0(t-T)}\right][1 + h(t)]$$

Since $\frac{l\sigma_r\theta_4}{\sigma_B\beta z^2} > 0$, $\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) > 0, h(t) > 0$ *and* $e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_3{}^*}{\partial z} = -\left[\frac{l\sigma_r\theta_4}{\sigma_B\beta z^2}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)] < 0$$

Hence $\frac{\partial \mu_3{}^*}{\partial z} < 0$

(ii) $\mu_3{}^* = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right)\right][1+h(t)]$

$$\frac{\partial \mu_3{}^*}{\partial \beta} = -\left[\frac{l\sigma_r\theta_4}{\sigma_B\beta^2 z}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)]$$

Since $\frac{l\sigma_r\theta_4}{\sigma_B\beta^2 z} > 0,\ \left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) > 0, h(t) > 0\ and\ e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_3{}^*}{\partial \beta} = -\left[\frac{l\sigma_r\theta_4}{\sigma_B\beta^2 z}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)] < 0$$

Hence $\frac{\partial \mu_3{}^*}{\partial \beta} < 0$

(iii) $\mu_3{}^* = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right)\right][1+h(t)]$

$$\frac{\partial \mu_3{}^*}{\partial l} = \left[\frac{\sigma_r\theta_4}{\sigma_B\beta z}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)]$$

Since $\frac{\sigma_r\theta_4}{\sigma_B\beta z} > 0,\ \left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) > 0, h(t) > 0\ and\ e^{\varphi_0(t-T)} > 0$

Then

$$\frac{\partial \mu_3{}^*}{\partial l} = \left[\frac{\sigma_r\theta_4}{\sigma_B\beta z}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)] > 0$$

Hence $\frac{\partial \mu_3{}^*}{\partial l} > 0$

(iii) $\mu_3{}^* = \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right)\right][1+h(t)]$

Assume that $(n_2 - n_1n_3) > 0$ and

$$\varphi_0 = n_3\lambda_1\sigma_s + \lambda_2 n_2(c_1 r\ +\ c_2) + 2n_3^2\sigma_s^2 + r - \mu_L$$

$$\sigma_r^2 = c_1 r\ +\ c_2$$

$$\frac{\partial \mu_3{}^*}{\partial r} = \frac{c_1}{2\sigma_B(c_1 r +\ c_2)^{\frac{1}{2}}}\left[(n_2 - n_1n_3) + \frac{l\theta_4}{\beta z}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\varphi_0(t-T)}\right][1+h(t)]$$

$$+\frac{l\theta_4(c_1 r +\ c_2)^{\frac{1}{2}}}{\beta z\sigma_B}\left[(\lambda_2 n_2 c_1\ +\ 1)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)\right][1+h(t)]e^{\varphi_0(t-T)}$$

Since $\frac{\sigma_r\theta_4}{\sigma_B\beta z} > 0,\ \left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) > 0, h(t) > 0\ and\ e^{\varphi_0(t-T)} > 0, (c_1 r +\ c_2)^{\frac{1}{2}} > 0$

Then

$$\frac{\partial \mu_3{}^*}{\partial r} = \frac{c_1}{2\sigma_B(c_1 r + c_2)^{\frac{1}{2}}}\left[(n_2 - n_1 n_3) + \frac{l\theta_4}{\beta z}\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}\right][1 + h(t)]$$
$$+ \frac{l\theta_4(c_1 r + c_2)^{\frac{1}{2}}}{\beta z \sigma_B}\left[(\lambda_2 n_2 c_1 + 1)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)\right][1 + h(t)]e^{\varphi_0(t-T)} > 0$$

Hence $\frac{\partial \mu_3{}^*}{\partial r} > 0$

Lemma 4.3 Suppose $z > 0, \theta_4 > 0,\ \beta > 0, \varphi_0 > 0, t \in [0,T], \vartheta > 0, \vartheta_0 > 0, n_1 > 0, n_2 > 0, n_3 > 0, l > 0\ \sigma_s > 0, \lambda_1 > 0$ and $\vartheta - \vartheta_0 - t > \vartheta - \vartheta_0 - T$ then

(i) $\frac{\partial \mu_1{}^*}{\partial z} < 0$ (ii) $\frac{\partial \mu_1{}^*}{\partial \beta} < 0$ (iii) $\frac{\partial \mu_1{}^*}{\partial l} > 0$ (iv) $\frac{\partial \mu_1{}^*}{\partial r} > 0$

$$\mu_1{}^* = 1 - \left(n_3 + l\left(\frac{\lambda_1 + n_3\sigma_s^2}{\beta z \sigma_s}\right)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}\right) - \frac{\sigma_r}{\sigma_B}\left[(n_2 - n_1 n_3) + \frac{l\theta_4}{\beta z}\left(\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\varphi_0(t-T)}\right)\right][1 + h(t)]$$

(i) $\mu_1{}^* = 1 - \mu_2{}^* - \mu_3{}^*$

$$\frac{\partial \mu_1{}^*}{\partial z} = -\frac{\partial \mu_2{}^*}{\partial z} - \frac{\partial \mu_3{}^*}{\partial z} = -\left(\frac{\partial \mu_2{}^*}{\partial z} + \frac{\partial \mu_3{}^*}{\partial z}\right)$$

But $\frac{\partial \mu_2{}^*}{\partial z} < 0$ and $\frac{\partial \mu_3{}^*}{\partial z} < 0$, therefore $\frac{\partial \mu_2{}^*}{\partial z} + \frac{\partial \mu_3{}^*}{\partial z} < 0$. Hence

$$\frac{\partial \mu_2{}^*}{\partial z} = -\left(\frac{\partial \mu_2{}^*}{\partial z} + \frac{\partial \mu_3{}^*}{\partial z}\right) > 0$$

(ii) $\frac{\partial \mu_1{}^*}{\partial \beta} = -\frac{\partial \mu_2{}^*}{\partial \beta} - \frac{\partial \mu_3{}^*}{\partial \beta} = -\left(\frac{\partial \mu_2{}^*}{\partial \beta} + \frac{\partial \mu_3{}^*}{\partial \beta}\right)$

Since $\frac{\partial \mu_2{}^*}{\partial \beta} < 0$ and $\frac{\partial \mu_2{}^*}{\partial \beta} < 0$, such that $\frac{\partial \mu_2{}^*}{\partial \beta} + \frac{\partial \mu_3{}^*}{\partial \beta} < 0$. Therefore

$$\frac{\partial \mu_1{}^*}{\partial \beta} = -\left(\frac{\partial \mu_2{}^*}{\partial \beta} + \frac{\partial \mu_3{}^*}{\partial \beta}\right) > 0$$

(iii) $\frac{\partial \mu_1{}^*}{\partial l} = -\frac{\partial \mu_2{}^*}{\partial l} - \frac{\partial \mu_3{}^*}{\partial l} = -\left(\frac{\partial \mu_2{}^*}{\partial l} + \frac{\partial \mu_3{}^*}{\partial l}\right)$

Since $\frac{\partial \mu_2{}^*}{\partial l} > 0$ and $\frac{\partial \mu_3{}^*}{\partial l} > 0$, such that $\frac{\partial \mu_2{}^*}{\partial l} + \frac{\partial \mu_3{}^*}{\partial l} > 0$. Therefore

$$\frac{\partial \mu_1{}^*}{\partial l} = -\left(\frac{\partial \mu_2{}^*}{\partial l} + \frac{\partial \mu_3{}^*}{\partial l}\right) < 0$$

(iii) $\frac{\partial \mu_1{}^*}{\partial r} = -\frac{\partial \mu_2{}^*}{\partial r} - \frac{\partial \mu_2{}^*}{\partial r} = -\left(\frac{\partial \mu_2{}^*}{\partial r} + \frac{\partial \mu_3{}^*}{\partial r}\right)$

Since $\frac{\partial \mu_2{}^*}{\partial r} > 0$ and $\frac{\partial \mu_3{}^*}{\partial r} > 0$, such that $\frac{\partial \mu_2{}^*}{\partial r} + \frac{\partial \mu_3{}^*}{\partial r} > 0$.

$$\frac{\partial \mu_1{}^*}{\partial r} = -\left(\frac{\partial \mu_2{}^*}{\partial r} + \frac{\partial \mu_3{}^*}{\partial r}\right) < 0$$

Therefore

$$\frac{\partial \mu_1{}^*}{\partial r} < 0$$

5 Discussion

Lemma 4.1 reveals that the optimal investment plan for stock decreases as the risk aversion coefficient increases; this implies that if other component of the optimal investment plan are kept constant, the fund manager will invest less fund in stock as risk aversion coefficient increases and this is a similar to lemma 4.2 which shows that investment in bond follows the same trend. Also, we observed from lemma 4.1 and 4.2 that as the initial wealth decreases, the fraction on the pension wealth invested in stock and bond increases. Also we observed that investment in stock and bond increases monotonically with respect to member's salary and risk free interest rate. From lemma 4.3, we observed that the optimal investment plan for the risk free asset increases with increase in risk aversion coefficient and initial wealth and decreases with increase in salary and interest rate.

In conclusion, we observed that since the risk free interest is stochastic, fund managers will prefer to invest more in stock and bond when the risk free interest rate is high and invest less in risk free asset and vice versa. This is contrary to the case where the risk free interest is not stochastic. This is simply because with the stochastic nature of the risk free interest rate, members may not be able to predict when the interest rate will decrease or increase.

6 Conclusion

Here we studied the optimal investment plan for a DC plan with refund of contributions, stochastic salary and affine interest rate model. Investment in a risk free (cash) and two risky assets (stock and zero-coupon bond) where the prices of the risky assets are modelled via geometric Brownian motions and the risk free interest rate is of affine structure. We obtained explicit solutions of the optimal investment plan for the three assets using mean variance utility function and presented theoretical analyses of the impact of some sensitive parameters on the optimal investment plan.

Competing Interests

Authors have declared that no competing interests exist.

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