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On Generalized 2-primes Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we introduce the generalized 2-primes sequences and we deal with, in detail, three special cases which we call them 2-primes, Lucas 2-primes and modified 2-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: 2-primes numbers; Lucas 2-primes numbers; generalized Fibonacci numbers.

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1 INTRODUCTION

In this paper, we investigate the generalized 2-primes sequences and we investigate, in detail, three special cases which we call them 2-primes, Lucas 2-primes and modified 2-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2, \quad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \;\; n \geq 2, \quad L_0 = 2, \; L_1 = 1$$

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respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

The generalized Fibonacci sequence $\{W_n(W_0, W_1; r, s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [1]) as follows:

$$W_n = rW_{n-1}+sW_{n-2}, \quad W_0 = a, W_1 = b, \ n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these numbers $\{W_n(a, b; r, s)\}$ are called Horadam numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	$W_n(a, b; r, s)$	Binet Formula	OEIS[5]
Fibonacci	$W_n(0, 1; 1, 1) = F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2, 1; 1, 1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0, 1; 2, 1) = P_n$	$\frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2, 2; 2, 1) = Q_n$	$\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$	A002203
Jacobsthal	$W_n(0, 1; 1, 2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2, 1; 1, 2) = j_n$	$2^n + (-1)^n$	A014551

Here OEIS stands for On-line Encyclopedia of Integer Sequences.

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [6,7,8,9,10,11,12,13,14,15,16,17,18,19].

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [20,21,22,23,24,25,26,27]. For higher order Pell sequences, see [28,29,30,31,32,33].

We can list some important properties of Horadam numbers that are needed.

- In 1843, Binet gave a formula which is called "Binet formula" for the usual Fibonacci numbers F_n by using the roots $\alpha_F = \frac{1+\sqrt{5}}{2}$, $\beta_F = \frac{1-\sqrt{5}}{2}$ of the characteristic equation $x^2 - x - 1 = 0$:

  $$F_n = \frac{\alpha_F^n - \beta_F^n}{\alpha - \beta}.$$

  Here α_F is called Golden Proportion (or Golden Number or Golden Section) (for details, see for example [34,35,36]).

  Binet formula of Horadam sequence can be calculated using its characteristic equation which is given as

  $$x^2 - rx - s = 0.$$

  The roots of characteristic equation are

  $$\alpha = \frac{r + \sqrt{\Delta}}{2}, \ \ \beta = \frac{r - \sqrt{\Delta}}{2}.$$

where $\Delta = r^2 + 4s$ and the followings hold

$$\begin{aligned} \alpha + \beta &= r \\ \alpha\beta &= -s. \end{aligned}$$

Using these roots and the recurrence relation, Binet formula can be given as follows

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.2}$$

where $A = b - a\beta$ and $B = b - a\alpha$. The Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n,

- The generating function for Horadam numbers is

$$g(x) = \frac{W_0 + (W_1 - rW_0)\,x}{1 - rx - sx^2}. \tag{1.3}$$

- The Cassini identity for Horadam numbers is

$$W_{n+1}W_{n-1} - W_n^2 = s^{n-1}(rW_0W_1 - W_1^2 - W_0^2 s). \tag{1.4}$$

A search of the literature turns up that there are many identities including Simson (Cassini), Catalan, d'Ocagne, Melham, Tagiuri, Gelin-Cesaro, Gould identities, see for example, [37,38,39,-40,41,42, 43,44].

- A summation formula for Horadam numbers is

$$\sum_{i=0}^{n} W_i = \frac{W_1 - W_0(r-1) + sW_n - W_{n+1}}{1 - r - s}. \tag{1.5}$$

- For $\Delta = r^2 + 4s > 0$, α and β are reals and $\alpha \neq \beta$. Note also that

$$\alpha^2 = \alpha\sqrt{\Delta} - s \tag{1.6}$$

and

$$\beta^2 = -\beta\sqrt{\Delta} - s. \tag{1.7}$$

-

$$\begin{aligned} A\alpha^n &= \alpha W_n + sW_{n-1}, \\ B\beta^n &= \beta W_n + sW_{n-1}. \end{aligned}$$

In this paper we consider the case $r = 2$, $s = 3$ and in this case we write $V_n = W_n$. A generalized 2-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} \tag{1.8}$$

with the initial values $V_0 = c_0, V_1 = c_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{2}{3}V_{-(n-1)} + \frac{1}{3}V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.8) holds for all integer n.

Eq. (1.2) can be used to obtain Binet formula of generalized 2-primes numbers. Binet formula of generalized 2-primes numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha - \beta)} + \frac{b_2\beta^n}{(\beta - \alpha)} = \frac{b_1\alpha^n - b_2\beta^n}{(\alpha - \beta)}$$

where

$$b_1 = V_1 - \beta V_0,\ b_2 = V_1 - \alpha V_0. \tag{1.9}$$

Here, α and β are the roots of the quadratic equation $x^2 - 2x - 3 = 0$. Moreover

$$\begin{aligned} \alpha &= 3 \\ \beta &= -1 \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta &= 2, \\ \alpha\beta &= -3, \\ \alpha - \beta &= 4. \end{aligned}$$

So

$$V_n = \frac{(V_1 + V_0)3^n - (V_1 - 3V_0)(-1)^n}{4}.$$

The first few generalized 2-primes numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized 2-primes numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{3}V_1 - \frac{2}{3}V_0$
2	$3V_0 + 2V_1$	$\frac{7}{9}V_0 - \frac{2}{9}V_1$
3	$6V_0 + 7V_1$	$\frac{7}{27}V_1 - \frac{20}{27}V_0$
4	$21V_0 + 20V_1$	$\frac{61}{81}V_0 - \frac{20}{81}V_1$
5	$60V_0 + 61V_1$	$\frac{61}{243}V_1 - \frac{182}{243}V_0$
6	$183V_0 + 182V_1$	$\frac{547}{729}V_0 - \frac{182}{729}V_1$
7	$546V_0 + 547V_1$	$\frac{547}{2187}V_1 - \frac{1640}{2187}V_0$
8	$1641V_0 + 1640V_1$	$\frac{4921}{6561}V_0 - \frac{1640}{6561}V_1$

Now we define three special cases of the sequence $\{V_n\}$. 2-primes sequence$\{G_n\}_{n\geq 0}$, Lucas 2-primes sequence $\{H_n\}_{n\geq 0}$ and modified 2-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = 2G_{n+1} + 3G_n, \quad G_0 = 1, G_1 = 2, \tag{1.10}$$

$$H_{n+2} = 2H_{n+1} + 3H_n, \quad H_0 = 2, H_1 = 2, \tag{1.11}$$

and

$$E_{n+2} = 2E_{n+1} + 3E_n, \quad E_0 = 1, E_1 = 1, \tag{1.12}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{2}{3}G_{-(n-1)} + \frac{1}{3}G_{-(n-2)}, \tag{1.13}$$

$$H_{-n} = -\frac{2}{3}H_{-(n-1)} + \frac{1}{3}H_{-(n-2)}, \tag{1.14}$$

and

$$E_{-n} = -\frac{2}{3}E_{-(n-1)} + \frac{1}{3}E_{-(n-2)}, \tag{1.15}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.13), (1.14) and (1.15) hold for all integer n.

Note that the sequences $\{G_n\}$, $\{H_n\}$ and $\{E_n\}$ are not indexed in [5] yet. Next, we present the first few values of the 2-primes, Lucas 2-primes and modified 2-primes numbers with positive and negative subscripts:

Table 3. The first few values of the special second-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12
G_n	1	2	7	20	61	182	547	1640	4921	14762	44287	132860	398581
G_{-n}		0	$\frac{1}{3}$	$-\frac{2}{9}$	$\frac{7}{27}$	$-\frac{20}{81}$	$\frac{61}{243}$	$-\frac{182}{729}$	$\frac{547}{2187}$	$-\frac{1640}{6561}$	$\frac{4921}{19683}$	$-\frac{14762}{59049}$	$\frac{44287}{177147}$
H_n	2	2	10	26	82	242	730	2186	6562	19682	59050	177146	531442
H_{-n}		$-\frac{2}{3}$	$\frac{10}{9}$	$-\frac{26}{27}$	$\frac{82}{81}$	$-\frac{242}{243}$	$\frac{730}{729}$	$-\frac{2186}{2187}$	$\frac{6562}{6561}$	$-\frac{19682}{19683}$	$\frac{59050}{59049}$	$-\frac{177146}{177147}$	$\frac{531442}{531441}$
E_n	1	1	5	13	41	121	365	1093	3281	9841	29525	88573	265721
E_{-n}		$-\frac{1}{3}$	$\frac{5}{9}$	$-\frac{13}{27}$	$\frac{41}{81}$	$-\frac{121}{243}$	$\frac{365}{729}$	$-\frac{1093}{2187}$	$\frac{3281}{6561}$	$-\frac{9841}{19683}$	$\frac{29525}{59049}$	$-\frac{88573}{177147}$	$\frac{265721}{531441}$

For all integers $n,$ 2-primes, Lucas 2-primes and modified 2-primes numbers (using initial conditions in (1.9)) can be expressed using Binet's formulas as

$$G_n = \frac{\alpha^{n+1}}{(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\alpha)} = \frac{3^{n+1}+(-1)^n}{4},$$

and

$$H_n = \alpha^n + \beta^n = 3^n + (-1)^n \,,$$

and

$$E_n = \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)} = \frac{3^n+(-1)^n}{2},$$

respectively.

Note that for all n we have

$$\begin{aligned} H_n &= 2E_n, \\ E_n &= G_n - G_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_{-n} &= \frac{1}{3^{n-1}}(-1)^n G_{n-2}, \;\; n \geq 2 \\ H_{-n} &= \frac{1}{3^n}(-1)^n H_n, \;\; n \geq 1, \\ E_{-n} &= \frac{1}{3^n}(-1)^n E_n, \;\; n \geq 1. \end{aligned}$$

2 GENERATING FUNCTIONS

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence $\{V_n\}.$

Lemma 2.1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized 2-primes sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x}{1-2x-3x^2}. \tag{2.1}$$

Proof. Using the definition of generalized 2-primes numbers, and substracting $2x\sum_{n=0}^{\infty}V_nx^n$ and $3x^2\sum_{n=0}^{\infty}V_nx^n$ from $\sum_{n=0}^{\infty}V_nx^n$ we obtain

$$\begin{aligned}(1-2x-3x^2)\sum_{n=0}^{\infty}V_nx^n &= \sum_{n=0}^{\infty}V_nx^n-2x\sum_{n=0}^{\infty}V_nx^n-3x^2\sum_{n=0}^{\infty}V_nx^n \\ &= \sum_{n=0}^{\infty}V_nx^n-2\sum_{n=0}^{\infty}V_nx^{n+1}-3\sum_{n=0}^{\infty}V_nx^{n+2} \\ &= \sum_{n=0}^{\infty}V_nx^n-2\sum_{n=1}^{\infty}V_{n-1}x^n-3\sum_{n=2}^{\infty}V_{n-2}x^n \\ &= (V_0+V_1x)-2V_0x+\sum_{n=2}^{\infty}(V_n-2V_{n-1}-3V_{n-2})x^n \\ &= V_0+(V_1-2V_0)x.\end{aligned}$$

Rearranging above equation, we obtain (2.1).

The previous Lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of 2-primes, Lucas 2-primes and modified 2-primes numbers are*

$$\sum_{n=0}^{\infty}G_nx^n=\frac{1}{1-2x-3x^2},$$

and

$$\sum_{n=0}^{\infty}H_nx^n=\frac{2-2x}{1-2x-3x^2},$$

and

$$\sum_{n=0}^{\infty}E_nx^n=\frac{1-x}{1-2x-3x^2},$$

respectively.

Proof. In Lemma 2.1, take $V_n=G_n$ with $G_0=1,G_1=2,$ $V_n=H_n$ with $H_0=2,H_1=2$ and $V_n=E_n$ with $E_0=1,E_1=1,$ respectively.

3 OBTAINING BINET FORMULA FROM GENERATING FUNCTION

We next find Binet formula of generalized 2-primes numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1. *(Binet formula of generalized 2-primes numbers)*

$$V_n=\frac{d_1\alpha^n}{(\alpha-\beta)}+\frac{d_2\beta^n}{(\beta-\alpha)} \tag{3.1}$$

where

$$\begin{aligned}d_1 &= V_0\alpha+(V_1-2V_0),\\ d_2 &= V_0\beta+(V_1-2V_0)\beta.\end{aligned}$$

Proof. Let

$$h(x) = 1 - 2x - 3x^2.$$

Then for some α and β we write

$$h(x) = (1 - \alpha x)(1 - \beta x)$$

i.e.,

$$1 - 2x - 3x^2 = (1 - \alpha x)(1 - \beta x). \quad (3.2)$$

Hence $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of $h(x)$. This gives α and β as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{3}{x^2} = 0.$$

This implies $x^2 - 2x - 3 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x}{(1 - \alpha x)(1 - \beta x)}.$$

Then we write

$$\frac{V_0 + (V_1 - 2V_0)x}{(1 - \alpha x)(1 - \beta x)} = \frac{A_1}{(1 - \alpha x)} + \frac{A_2}{(1 - \beta x)}. \quad (3.3)$$

So

$$V_0 + (V_1 - 2V_0)x = A_1(1 - \beta x) + A_2(1 - \alpha x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - 2V_0)\frac{1}{\alpha} = A_1(1 - \frac{\beta}{\alpha})$. This gives

$$A_1 = \frac{\alpha(V_0 + (V_1 - 2V_0)\frac{1}{\alpha})}{(\alpha - \beta)} = \frac{V_0\alpha + (V_1 - 2V_0)}{(\alpha - \beta)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta + (V_1 - 2V_0)\beta}{(\beta - \alpha)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1 - \alpha x)^{-1} + A_2(1 - \beta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha + (V_1 - 2V_0)}{(\alpha - \beta)}, \\ A_2 &= \frac{V_0\beta + (V_1 - 2V_0)\beta}{(\beta - \alpha)}, \end{aligned}$$

and then we get (3.1).

Note that from (1.9) and (3.1) we have

$$\begin{aligned} V_1-\beta V_0 &= V_0\alpha+(V_1-2V_0), \\ V_1-\alpha V_0 &= V_0\beta+(V_1-2V_0)\beta. \end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of 2-primes, Lucas 2-primes and modified 2-primes sequences.

Corollary 3.2. *Binet formulas of 2-primes, Lucas 2-primes and modified 2-primes sequences are*

$$G_n=\frac{\alpha^{n+1}}{(\alpha-\beta)}+\frac{\beta^{n+1}}{(\beta-\alpha)}=\frac{3^{n+1}+(-1)^n}{4},$$

and

$$H_n=\alpha^n+\beta^n=3^n+(-1)^n\,,$$

and

$$E_n=\frac{(\alpha-1)\alpha^n}{(\alpha-\beta)}+\frac{(\beta-1)\beta^n}{(\beta-\alpha)}=\frac{3^n+(-1)^n}{2},$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [29]. Take $k=i=2$ in Corollary 3.1 in [29]. Let

$$\Lambda=\begin{pmatrix}\alpha & 1\\ \beta & 1\end{pmatrix}, \Lambda_1=\begin{pmatrix}\alpha^{n-1} & 1\\ \beta^{n-1} & 1\end{pmatrix}, \Lambda_2=\begin{pmatrix}\alpha & \alpha^{n-1}\\ \beta & \beta^{n-1}\end{pmatrix}.$$

Then the Binet formula for 2-primes numbers is

$$\begin{aligned} G_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{2}G_{3-j}\det(\Lambda_j)=\frac{1}{\Lambda}(G_2\det(\Lambda_1)+G_1\det(\Lambda_2)) \\ &= \frac{1}{\det(\Lambda)}(7\det(\Lambda_1)+2\det(\Lambda_2)) \\ &= \left(7\begin{vmatrix}\alpha^{n-1} & 1\\ \beta^{n-1} & 1\end{vmatrix}+2\begin{vmatrix}\alpha & \alpha^{n-1}\\ \beta & \beta^{n-1}\end{vmatrix}\right)/\begin{vmatrix}\alpha & 1\\ \beta & 1\end{vmatrix}. \end{aligned}$$

Similarly, we obtain the Binet formula for Lucas 2-primes and modified 2-primes numbers as

$$\begin{aligned} H_n &= \frac{1}{\Lambda}(H_2\det(\Lambda_1)+H_1\det(\Lambda_2)) \\ &= \left(10\begin{vmatrix}\alpha^{n-1} & 1\\ \beta^{n-1} & 1\end{vmatrix}+2\begin{vmatrix}\alpha & \alpha^{n-1}\\ \beta & \beta^{n-1}\end{vmatrix}\right)/\begin{vmatrix}\alpha & 1\\ \beta & 1\end{vmatrix}, \end{aligned}$$

and

$$\begin{aligned} E_n &= \frac{1}{\Lambda}(E_2\det(\Lambda_1)+E_1\det(\Lambda_2)) \\ &= \left(5\begin{vmatrix}\alpha^{n-1} & 1\\ \beta^{n-1} & 1\end{vmatrix}+\begin{vmatrix}\alpha & \alpha^{n-1}\\ \beta & \beta^{n-1}\end{vmatrix}\right)/\begin{vmatrix}\alpha & 1\\ \beta & 1\end{vmatrix}, \end{aligned}$$

respectively.

4 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 2-primes sequence $\{V_n\}_{n\geq 0}$.

Theorem 4.1 (Simson Formula of Generalized 2-Primes Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{vmatrix} = (-1)^n 3^n \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \tag{4.1}$$

Proof. Eq. (4.1) is given in Soykan [45].

The previous theorem gives the following results as particular examples.

Corollary 4.2. *For all integers n, Simson formula of 2-primes, Lucas 2-primes and modified 2-primes numbers are given as*

$$\begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} = (-1)^{n+1} 3^n,$$

and

$$\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} = 16(-1)^{n+1} 3^{n-1},$$

and

$$\begin{vmatrix} E_{n+1} & E_n \\ E_n & E_{n-1} \end{vmatrix} = 4(-1)^{n+1} 3^{n-1},$$

respectively.

5 SOME IDENTITIES

In this section, we obtain some identities of 2-primes, Lucas 2-primes and modified 2-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1. *The following equalities are true:*

$$\begin{aligned} 81H_n &= 82G_{n+4} - 242G_{n+3}, \\ 27H_n &= -26G_{n+3} + 82G_{n+2}, \\ 9H_n &= 10G_{n+2} - 26G_{n+1}, \\ 3H_n &= -2G_{n+1} + 10G_n, \\ H_n &= 2G_n - 2G_{n-1}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} 72G_n &= 5H_{n+4} - 13H_{n+3}, \\ 24G_n &= -H_{n+3} + 5H_{n+2}, \\ 8G_n &= H_{n+2} - H_{n+1}, \\ 8G_n &= H_{n+1} + 3H_n, \\ 8G_n &= 5H_n + 3H_{n-1}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$H_n = a \times G_{n+4} + b \times G_{n+3}$$

and solving the system of equations

$$\begin{aligned} H_0 &= a \times G_4 + b \times G_3 \\ H_1 &= a \times G_5 + b \times G_4 \end{aligned}$$

we find that $a = \frac{82}{81}, b = -\frac{242}{81}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned} 81E_n &= 41G_{n+4} - 121G_{n+3}, \\ 27E_n &= -13G_{n+3} + 41G_{n+2}, \\ 9E_n &= 5G_{n+2} - 13G_{n+1}, \\ 3E_n &= -G_{n+1} + 5G_n, \\ E_n &= G_n - G_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 36G_n &= 5E_{n+4} - 13E_{n+3}, \\ 12G_n &= -E_{n+3} + 5E_{n+2}, \\ 4G_n &= E_{n+2} - E_{n+1}, \\ 4G_n &= E_{n+1} + 3E_n, \\ 4G_n &= 5E_n + 3E_{n-1}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned} 27H_n &= 14E_{n+4} - 40E_{n+3}, \\ 9H_n &= -4E_{n+3} + 14E_{n+2}, \\ 3H_n &= 2E_{n+2} - 4E_{n+1}, \\ H_n &= 2E_n, \end{aligned}$$

and

$$\begin{aligned} 54E_n &= 7H_{n+4} - 20H_{n+3} \\ 18E_n &= -2H_{n+3} + 7H_{n+2} \\ 6E_n &= H_{n+2} - 2H_{n+1} \\ 2E_n &= H_n \end{aligned}$$

We now present a few special identities for the generalized 2-primes sequence $\{V_n\}$.

Theorem 5.4. *(Catalan's identity of the generalized 2-primes sequence) For all integers n and m, the following identity holds:*

$$V_{n+m}V_{n-m} - V_n^2 = \frac{1}{16}3^{n-m}(-1)^{n-m}(3V_0 - V_1)(V_0 + V_1)(3^m - (-1)^m)^2 .$$

Proof. We use the identity
$$V_n = \frac{(V_1+V_0)3^n - (V_1-3V_0)(-1)^n}{4}.$$

As special cases of the above theorem, we have the following corollary.

Corollary 5.5. *For all integers n and m, the following identities hold:*

(a) $G_{n+m}G_{n-m} - G_n^2 = \frac{1}{16}3^{n-m+1}(-1)^{n-m}(3^m-(-1)^m)^2$.

(b) $H_{n+m}H_{n-m} - H_n^2 = 3^{n-m}(-1)^{n-m}(3^m-(-1)^m)^2$.

(c) $E_{n+m}E_{n-m} - E_n^2 = \frac{1}{4}3^{n-m}(-1)^{n-m}(3^m-(-1)^m)^2$.

Note that for $m = 1$ in Catalan's identity of the generalized 2-primes sequence, we get the Cassini identity for the generalized 2-primes sequnce.

Theorem 5.6. *(Cassini's identity of the generalized 2-primes sequence) For all integers n and m, the following identity holds:*
$$V_{n+1}V_{n-1} - V_n^2 = 3^{n-1}(-1)^{n-1}(3V_0-V_1)(V_0+V_1).$$

As special cases of the above theorem, we have the following corollary.

Corollary 5.7. *For all integers n and m, the following identities hold:*

(a) $G_{n+1}G_{n-1} - G_n^2 = 3^n(-1)^{n-1}$.

(b) $H_{n+1}H_{n-1} - H_n^2 = 16\times 3^{n-1}(-1)^{n-1}$.

(c) $E_{n+1}E_{n-1} - E_n^2 = 4\times 3^{n-1}(-1)^{n-1}$.

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using
$$V_n = \frac{(V_1+V_0)3^n - (V_1-3V_0)(-1)^n}{4}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of generalized 2-primes sequence $\{V_n\}$.

Theorem 5.8. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*
$$V_{m+1}V_n - V_mV_{n+1} = \frac{1}{4}(V_1-3V_0)(V_0+V_1)((-1)^m 3^n - (-1)^n 3^m).$$

(b) *(Gelin-Cesàro's identity)*
$$\begin{aligned} V_{n+2}V_{n+1}V_{n-1}V_{n-2} - V_n^4 &= \frac{1}{432}(-3)^n(3V_0-V_1)(V_0+V_1)((3+3^{2n+1}+58(-3)^n)V_1^2 \\ &\quad +3(9+3^{2n}-58(-1)^n3^n)V_0^2 - 2(9-3^{2n+1}+58(-3)^n)V_1V_0). \end{aligned}$$

(c) *(Melham's identity)*
$$\begin{aligned} V_{n+1}V_{n+2}V_{n+6} - V_{n+3}^3 &= \frac{3}{4}(-1)^n3^n(V_1-3V_0)(V_0+V_1)((45\times 3^n - 17(-1)^n)V_1 \\ &\quad +3(15\times 3^n + 17(-1)^n)V_0). \end{aligned}$$

Proof. Use the identity $V_n = \frac{(V_1+V_0)3^n-(V_1-3V_0)(-1)^n}{4}$.

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of 2-primes sequence $\{G_n\}$.

Corollary 5.9. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$G_{m+1}G_n - G_mG_{n+1} = \frac{1}{4}\left((-1)^n 3^{m+1} - (-1)^m 3^{n+1}\right).$$

(b) *(Gelin-Cesàro's identity)*

$$G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = \frac{1}{16}((-1)^n 3^{n-1} + 9(-1)^n 3^{3n-1} - 58 \times 3^{2n-1}).$$

(c) *(Melham's identity)*

$$G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = \frac{1}{4}(-17 \times 3^{n+2} - 5(-1)^n 3^{2n+5}).$$

Second one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of Lucas 2-primes sequence $\{H_n\}$.

Corollary 5.10. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_mH_{n+1} = 4\left((-1)^n 3^m - (-1)^m 3^n\right).$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = 16(3^{2n+1} - 58(-1)^n 3^n + 3)(-1)^n 3^{n-3}.$$

(c) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = -16 \times 3^{n+1}(45(-1)^n 3^n + 17)).$$

Third one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified 2-primes sequence $\{E_n\}$.

Corollary 5.11. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (-1)^n 3^m - (-1)^m 3^n.$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (-1)^n (3^{3n-2} + 3^{n-2}) - 58 \times 3^{2n-3}.$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = -2 \times 3^{n+1}(45(-1)^n 3^n + 17).$$

6 SUMS

The following proposition presents some formulas of generalized 2-primes numbers with positive subscripts.

Proposition 6.1. *If $r = 2, s = 3$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{4}(V_{n+2} - V_{n+1} - V_1 + V_0)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{8}((2n+5)\,V_{2n+2} - 6\,(n+2)\,V_{2n+1} + 2V_1 - 7V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{8}(-2\,(n+1)\,V_{2n+2} + 3\,(2n+5)\,V_{2n+1} - 3V_1 + 6V_0)$.

Proof.

(a) Take $x = 1, r = 2, s = 3$ in Theorem 2.1 (a) in [46].

(b) We use Theorem 2.1 (b) in [46]. If we set $r = 2, s = 3$ in Theorem 2.1 (b) in [46] then we have

$$\sum_{k=0}^{n} x^k V_{2k} = \frac{-\,(3x-1)\,x^{n+1}V_{2n+2} + 6x^{n+2}V_{2n+1} - 2xV_1 + (7x-1)\,V_0}{-9x^2+10x-1}.$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} V_{2k} &= \left.\frac{\frac{d}{dx}(-\,(3x-1)\,x^{n+1}V_{2n+2} + 6x^{n+2}V_{2n+1} - 2xV_1 + (7x-1)\,V_0)}{\frac{d}{dx}(-9x^2+10x-1)}\right|_{x=1} \\ &= \frac{1}{8}((2n+5)\,V_{2n+2} - 6\,(n+2)\,V_{2n+1} + 2V_1 - 7V_0). \end{aligned}$$

(c) We use Theorem 2.1 (c) in [46]. If we set $r = 2, s = 3$ in Theorem 2.1 (c) in [46] then we have

$$\sum_{k=0}^{n} x^k V_{2k+1} = \frac{2x^{n+1}V_{2n+2} - 3\,(3x-1)\,x^{n+1}V_{2n+1} + (3x-1)\,V_1 - 6xV_0}{-9x^2+10x-1}.$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} V_{2k+1} &= \left.\frac{\frac{d}{dx}(2x^{n+1}V_{2n+2} - 3\,(3x-1)\,x^{n+1}V_{2n+1} + (3x-1)\,V_1 - 6xV_0)}{\frac{d}{dx}(-9x^2+10x-1)}\right|_{x=1} \\ &= \frac{1}{8}(-2\,(n+1)\,V_{2n+2} + 3\,(2n+5)\,V_{2n+1} - 3V_1 + 6V_0). \end{aligned}$$

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 2-primes numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 2$).

Corollary 6.1. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} G_k = \frac{1}{4}(G_{n+2} - G_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} G_{2k} = \frac{1}{8}((2n+5)\,G_{2n+2} - 6\,(n+2)\,G_{2n+1} - 3)$.

(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{8}(-2\,(n+1)\,G_{2n+2} + 3\,(2n+5)\,G_{2n+1})$.

Second one presents some summing formulas of Lucas 2-primes numbers (take $V_n = H_n$ with $H_0 = 2, H_1 = 2$).

Corollary 6.2. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} H_k = \frac{1}{4}(H_{n+2} - H_{n+1})$.

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{8}((2n+5) H_{2n+2} - 6(n+2) H_{2n+1} - 10)$.

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{8}(-2(n+1) H_{2n+2} + 3(2n+5) H_{2n+1} + 6)$.

Third one presents some summing formulas of modified 2-primes numbers (take $V_n = E_n$ with $E_0 = 1, E_1 = 1$).

Corollary 6.3. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} E_k = \frac{1}{4}(E_{n+2} - E_{n+1})$.

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{8}((2n+5) E_{2n+2} - 6(n+2) E_{2n+1} - 5)$.

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{8}(-2(n+1) E_{2n+2} + 3(2n+5) E_{2n+1} + 3)$.

The following proposition presents some formulas of generalized 2-primes numbers with negative subscripts.

Proposition 6.2. *If $r = 2, s = 3$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{4}(-5V_{-n-1} - 3V_{-n-2} + V_1 - V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{8}((2n+1) V_{-2n} - 6(n+1) V_{-2n-1} + 2V_1 - 5V_0)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{8}(-2(n+2) V_{-2n} + 3(2n+1) V_{-2n-1} - V_1 + 6V_0)$.

Proof.

(a) Take $x = 1, r = 2, s = 3$ in Theorem 3.1 (a) in [46].

(b) We use Theorem 3.1 (b) in [46]. If we set $r = 2, s = 3$ in Theorem 3.1 (b) in [46] then we have

$$\sum_{k=1}^{n} x^k V_{-2k} = \frac{-x^{n+1}(x-3) V_{-2n} - 6x^{n+1} V_{-2n-1} + 2xV_1 + x(x-7) V_0}{-x^2 + 10x - 9}.$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=1}^{n} V_{-2k} &= \left. \frac{\frac{d}{dx}(-x^{n+1}(x-3) V_{-2n} - 6x^{n+1} V_{-2n-1} + 2xV_1 + x(x-7) V_0)}{\frac{d}{dx}(-x^2 + 10x - 9)} \right|_{x=1} \\ &= \frac{1}{8}((2n+1) V_{-2n} - 6(n+1) V_{-2n-1} + 2V_1 - 5V_0). \end{aligned}$$

(c) We use Theorem 2.1 (c) in [46]. If we set $r = 2, s = 3$ in Theorem 2.1 (c) in [46] then we have

$$\sum_{k=1}^{n} x^k V_{-2k+1} = \frac{-2x^{n+2} V_{-2n} - 3(x-3) x^{n+1} V_{-2n-1} + x(x-3) V_1 + 6xV_0}{-x^2 + 10x - 9}.$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=1}^{n} V_{-2k+1} &= \left. \frac{\frac{d}{dx}(-2x^{n+2} V_{-2n} - 3(x-3) x^{n+1} V_{-2n-1} + x(x-3) V_1 + 6xV_0)}{\frac{d}{dx}(-x^2 + 10x - 9)} \right|_{x=1} \\ &= \frac{1}{8}(-2(n+2) V_{-2n} + 3(2n+1) V_{-2n-1} - V_1 + 6V_0). \end{aligned}$$

From the above proposition, we have the following corollary which gives sum formulas of 2-primes numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 2$).

Corollary 6.4. *For $n \geq 1$, 2-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n} G_{-k} = \frac{1}{4}(-5G_{-n-1} - 3G_{-n-2} + 1)$.

(b) $\sum_{k=1}^{n} G_{-2k} = \frac{1}{8}((2n+1) G_{-2n} - 6(n+1) G_{-2n-1} - 1)$.

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{8}(-2(n+2) G_{-2n} + 3(2n+1) G_{-2n-1} + 4)$.

Taking $V_n = H_n$ with $H_0 = 2, H_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of 2-primes -Lucas numbers.

Corollary 6.5. *For $n \geq 1$, 2-primes -Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} H_{-k} = \frac{1}{4}(-5H_{-n-1} - 3H_{-n-2})$.

(b) $\sum_{k=1}^{n} H_{-2k} = \frac{1}{8}((2n+1) H_{-2n} - 6(n+1) H_{-2n-1} - 6)$.

(c) $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{8}(-2(n+2) H_{-2n} + 3(2n+1) H_{-2n-1} + 10)$.

From the above proposition, we have the following corollary which gives sum formulas of modified 2-primes numbers (take $V_n = E_n$ with $E_0 = 1, E_1 = 1$).

Corollary 6.6. *For $n \geq 1$, modified 2-primes numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k} = \frac{1}{4}(-5E_{-n-1} - 3E_{-n-2})$.

(b) $\sum_{k=1}^{n} E_{-2k} = \frac{1}{8}((2n+1) E_{-2n} - 6(n+1) E_{-2n-1} - 3)$.

(c) $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{8}(-2(n+2) E_{-2n} + 3(2n+1) E_{-2n-1} + 5)$.

The following proposition presents some formulas of generalized 2-primes numbers with positive subscripts.

Proposition 6.3. *If $r = 2, s = 3$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k^2 = \frac{1}{32}((2n+7) V_{n+2}^2 + (18n+49) V_{n+1}^2 - 12(n+3) V_{n+2}V_{n+1} - 5V_1^2 - 31V_0^2 + 24V_1V_0)$.

(b) $\sum_{k=0}^{n} V_{k+1}V_k = \frac{1}{16}(-(n+2) V_{n+2}^2 - 9(n+3) V_{n+1}^2 + (6n+17) V_{n+2}V_{n+1} + V_1^2 + 18V_0^2 - 11V_1V_0)$.

Proof. The proof can be given using induction on n.

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 2-primes numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 2$).

Corollary 6.7. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{32}((2n+7) G_{n+2}^2 + (18n+49) G_{n+1}^2 - 12(n+3) G_{n+2}G_{n+1} - 3)$.

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{16}(-(n+2) G_{n+2}^2 - 9(n+3) G_{n+1}^2 + (6n+17) G_{n+2}G_{n+1})$.

Second one presents some summing formulas of Lucas 2-primes numbers (take $V_n = H_n$ with $H_0 = 2, H_1 = 2$).

Corollary 6.8. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{32}((2n+7) H_{n+2}^2 + (18n+49) H_{n+1}^2 - 12(n+3) H_{n+2}H_{n+1} - 48)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{16}(-(n+2) H_{n+2}^2 - 9(n+3) H_{n+1}^2 + (6n+17) H_{n+2}H_{n+1} + 32)$.

Third one presents some summing formulas of modified 2-primes numbers (take $V_n = E_n$ with $E_0 = 1, E_1 = 1$).

Corollary 6.9. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{32}((2n+7)E_{n+2}^2 + (18n+49)E_{n+1}^2 - 12(n+3)E_{n+2}E_{n+1} - 12)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{16}(-(n+2)E_{n+2}^2 - 9(n+3)E_{n+1}^2 + (6n+17)E_{n+2}E_{n+1} + 8)$.

The following proposition presents some formulas of generalized 2-primes numbers with negative subscripts.

Proposition 6.4. *If $r = 2, s = 3$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k}^2 = \frac{1}{32}((2n+1)V_{-n+1}^2 + (18n+23)V_{-n}^2 - 12(n+1)V_{-n+1}V_{-n} - V_1^2 - 23V_0^2 + 12V_1V_0)$.

(b) $\sum_{k=1}^{n} V_{-k+1}V_{-k} = \frac{1}{16}(-(n+2)V_{-n+1}^2 - 9(n+1)V_{-n}^2 + (6n+7)V_{-n+1}V_{-n} + 2V_1^2 + 9V_0^2 - 7V_1V_0)$.

Proof. The proof can be given using induction on n.

From the above proposition, we have the following corollary which gives sum formulas of 2-primes numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 2$).

Corollary 6.10. *For $n \geq 1$, 2-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} G_{-k}^2 = \frac{1}{32}((2n+1)G_{-n+1}^2 + (18n+23)G_{-n}^2 - 12(n+1)G_{-n+1}G_{-n} - 3)$.

(b) $\sum_{k=1}^{n} G_{-k+1}G_{-k} = \frac{1}{16}(-(n+2)G_{-n+1}^2 - 9(n+1)G_{-n}^2 + (6n+7)G_{-n+1}G_{-n} + 3)$.

Taking $V_n = H_n$ with $H_0 = 2, H_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of 2-primes -Lucas numbers.

Corollary 6.11. *For $n \geq 1$,*

(a) $\sum_{k=1}^{n} H_{-k}^2 = \frac{1}{32}((2n+1)H_{-n+1}^2 + (18n+23)H_{-n}^2 - 12(n+1)H_{-n+1}H_{-n} - 48)$.

(b) $\sum_{k=1}^{n} H_{-k+1}H_{-k} = \frac{1}{16}(-(n+2)H_{-n+1}^2 - 9(n+1)H_{-n}^2 + (6n+7)H_{-n+1}H_{-n} + 16)$.

From the above proposition, we have the following corollary which gives sum formulas of modified 2-primes numbers (take $V_n = E_n$ with $E_0 = 1, E_1 = 1$).

Corollary 6.12. *For $n \geq 1$, modified 2-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} E_{-k}^2 = \frac{1}{32}((2n+1)E_{-n+1}^2 + (18n+23)E_{-n}^2 - 12(n+1)E_{-n+1}E_{-n} - 12)$.

(b) $\sum_{k=1}^{n} E_{-k+1}E_{-k} = \frac{1}{16}(-(n+2)E_{-n+1}^2 - 9(n+1)E_{-n}^2 + (6n+7)E_{-n+1}E_{-n} + 4)$.

7 MATRICES RELATED WITH GENERALIZED 2-PRIMES NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [47]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & s \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}.$$

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

such that $\det A = -3$. From (1.8) we have

$$\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_n \\ V_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = G_n$ in (7.2) we have

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} G_n & 3G_{n-1} \\ G_{n-1} & 3G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_n & 3V_{n-1} \\ V_{n-1} & 3V_{n-2} \end{pmatrix}$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (7.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n-1} & 3V_{n-2} \\ V_{n-2} & 3V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} 2V_{n-1} + 3V_{n-2} & 6V_{n-2} + 9V_{n-3} \\ V_{n-1} & 3V_{n-2} \end{pmatrix} = C_n \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + 3A^{n-2}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-3)^n$$

for all integer m and n.

Theorem 7.2. *For $m, n \geq 0$ we have*

$$V_{n+m} = V_n G_m + 3V_{n-1}G_{m-1} \tag{7.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 7.1. By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n, (7.4) is true.

Corollary 7.3. *For all integers m, n, we have*

$$G_{n+m} = G_n G_m + 3G_{n-1}G_{m-1}, \tag{7.5}$$

$$H_{n+m} = H_n G_m + 3H_{n-1}G_{m-1}, \tag{7.6}$$

$$E_{n+m} = E_n G_m + 3E_{n-1}G_{m-1}. \tag{7.7}$$

8 CONCLUSIONS

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized generalized 2-primes sequence (it's three special cases, namely, 2-primes, Lucas 2-primes and modified 2-primes sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

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Author have declared that no competing interests exist.

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