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Research Article

The Maschke-Type Theorem and Morita Context for BiHom-Smash Products

Bingliang Shen[1] and Ling Liu[2]

[1]*Shanghai University of Finance & Economics Zhejiang College, Jinhua 321013, China*
[2]*College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321004, China*

Correspondence should be addressed to Ling Liu; ntliulin@zjnu.cn

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Let $(H, \alpha_H, \beta_H, \omega_H, \psi_H, S_H)$ be a BiHom-Hopf algebra and (A, α_A, β_A) be an (H, α_H, β_H)-module BiHom-algebra. Then, in this paper, we study some properties on the BiHom-smash product $A\#H$. We construct the Maschke-type theorem for the BiHom-smash product $A\#H$ and form an associated Morita context $[A^H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, A\#H]$.

1. Introduction

The first instance of Hom-type algebras appeared in the physics literature when looking for quantum deformations of some algebras of vector fields in 1990's, such as Witt and Virasoro algebras ([1, 2]). This kind of algebras obtained by deforming certain Lie algebras no longer satisfied the Jacobi identity, but a modified version of it involving a homomorphism. Such algebra (called Hom-Lie algebra) was given in [3, 4]. The associative counterpart of Hom-Lie algebras has been introduced in [5] (called Hom-associative algebras), and Hom-analogues of other algebraic structures have been introduced afterwards, Hom-coassociative coalgebras, Hom-bialgebras, Hom-pre-Lie algebras etc.

A categorical approach to Hom-type algebras was considered in [6]. A generalization has been given in [7], where a construction of a Hom-category including a group action led to concepts of BiHom-type algebras. Hence, BiHom-associative algebras and BiHom-Lie algebras, involving two linear maps (called structure maps), were introduced. The main tool to obtain examples of Hom-algebras from classical algebras, the so-called Yau twisting, works perfectly fine also in the BiHom-type case. There is a growing literature on Hom and BiHom-type algebras, and let us just mention the very recent papers [8–12].

Let H be a Hopf algebra and A an H-module algebra; then, as well known, we can construct the smash product algebra $A\#H$ (see [13] or [14]). Smash products plays an important role in the lifting method for the classification of finite-dimensional pointed Hopf algebras (see [15]). The Hom-forms of the smash product can be found in the following literature. In [16], the Maschke-type theorem for the Hom-smash product is given, and the Morita context is constructed. In [7], the authors defined the BiHom-smash product and gave some examples. Now, it is natural to ask how to prove the Maschke-type theorem and construct the associated Morita context for the BiHom-smash product?

The main aim of this paper is to give a positive answer to the above questions. We use the same strategy as in the Hom case and get an analogue of the Maschke-type theorem and form an associated Morita context between the BiHom-smash product and its BiHom-subalgebra in the setting of BiHom-Hopf algebras.

This paper is organized as follows. In Section 2, we recall some definitions and basic results related to BiHom-algebras, BiHom-coalgebras, BiHom-bialgebras, BiHom-modules, and module BiHom-algebras. In Sections 3 and 4, we study some properties on the BiHom-smash product $A\#H$. If $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ is a finite-dimensional semisimple BiHom-Hopf algebra, then we construct the Maschke-type theorem for the BiHom-smash product $A\#H$. We also prove that $[A^H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, A\#H]$ forms an associated Morita context, where $(A^H, \alpha_A|_{A^H}, \beta_A|_{A^H})$ is a BiHom-subalgebra of (H, α_H, β_H)-invariants in (A, α_A, β_A).

2. Preliminaries

We work over a base field $\Bbbk$. All algebras, linear spaces, etc. will be over $\Bbbk$; unadorned $\otimes$ means $\otimes_{\Bbbk}$. We use Sweedler's notation for terminologies on coalgebras. For a coalgebra C, we write comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$.

Definition 1 ([7]). A BiHom-associative algebra is a 4-tuple (A, μ, α, β), where A is a linear space and $\alpha, \beta : A \longrightarrow A$ and $\mu : A \otimes A \longrightarrow A$ are linear maps such that $\alpha \circ \beta = \beta \circ \alpha$, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, $\beta(x \cdot y) = \beta(x) \cdot \beta(y)$ and

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z), \tag{1}$$

for all $x, y, z \in A$. The maps α and β (in this order) are called the structure maps of A, and condition (1) is called the BiHom-associativity condition.

A morphism $f : (A, \mu_A, \alpha_A, \beta_A) \longrightarrow (B, \mu_B, \alpha_B, \beta_B)$ of BiHom-associative algebras is a linear map $f : A \longrightarrow B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$, and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

A BiHom-associative algebra (A, μ, α, β) is called unital if there exists an element $1_A \in A$ (called a unit) such that $\alpha(1_A) = 1_A, \beta(1_A) = 1_A$ and

$$a1_A = \alpha(a) \text{ and } 1_A a = \beta(a), \forall a \in A. \tag{2}$$

Definition 2 ([7]). A BiHom-coassociative coalgebra is a 4-tuple $(C, \Delta, \psi, \omega)$, in which C is a linear space, and $\psi, \omega : C \longrightarrow C$ and $\Delta : C \longrightarrow C \otimes C$ are linear maps, such that $\psi \circ \omega = \omega \circ \psi$, $(\psi \otimes \psi) \circ \Delta = \Delta \circ \psi$, $(\omega \otimes \omega) \circ \Delta = \Delta \circ \omega$, and

$$(\Delta \otimes \psi) \circ \Delta = (\omega \otimes \Delta) \circ \Delta. \tag{3}$$

The maps ψ and ω (in this order) are called the structure maps of C, and condition (3) is called the BiHom-coassociativity condition.

Let us record the formula expressing the BiHom-coassociativity of Δ:

$$\Delta(c_1) \otimes \psi(c_2) = \omega(c_1) \otimes \Delta(c_2), \forall c \in C. \tag{4}$$

A morphism $g : (C, \Delta_C, \psi_C, \omega_C) \longrightarrow (D, \Delta_D, \psi_D, \omega_D)$ of BiHom-coassociative coalgebras is a linear map $g : C \longrightarrow D$ such that $\psi_D \circ g = g \circ \psi_C$, $\omega_D \circ g = g \circ \omega_C$, and $(g \otimes g) \circ \Delta_C = \Delta_D \circ g$.

A BiHom-coassociative coalgebra $(C, \Delta, \psi, \omega)$ is called counital if there exists a linear map $\varepsilon : C \longrightarrow \Bbbk$ (called a counit) such that

$$\begin{gathered}\varepsilon \circ \psi = \varepsilon, \varepsilon \circ \omega = \varepsilon,\\ (id_C \otimes \varepsilon) \circ \Delta = \omega \text{ and } (\varepsilon \otimes id_C) \circ \Delta = \psi.\end{gathered} \tag{5}$$

Definition 3 ([7]). A BiHom-bialgebra is a 7-tuple $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$, with the property that (H, μ, α, β) is a BiHom-associative algebra, $(H, \Delta, \psi, \omega)$ is a BiHom-coassociative coalgebra and moreover, the following relations are satisfied, for all $h, h' \in H$:

$$\begin{aligned}&\Delta\left(hh'\right) = h_1 h'_1 \otimes h_2 h'_2,\\ &\alpha \circ \psi = \psi \circ \alpha, \alpha \circ \omega = \omega \circ \alpha, \beta \circ \psi = \psi \circ \beta, \beta \circ \omega = \omega \circ \beta,\\ &(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, (\beta \otimes \beta) \circ \Delta = \Delta \circ \beta,\\ &\psi\left(hh'\right) = \psi(h)\psi\left(h'\right), \omega\left(hh'\right) = \omega(h)\omega\left(h'\right).\end{aligned} \tag{6}$$

We say that H is a unital and counital BiHom-bialgebra if, in addition, it admits a unit 1_H and a counit ε_H such that

$$\begin{aligned}&\Delta(1_H) = 1_H \otimes 1_H, \varepsilon_H(1_H) = 1, \psi(1_H) = 1_H, \omega(1_H) = 1_H,\\ &\varepsilon_H \circ \alpha = \varepsilon_H, \varepsilon_H \circ \beta = \varepsilon_H, \varepsilon_H\left(hh'\right) = \varepsilon_H(h)\varepsilon_H\left(h'\right), \forall h, h' \in H.\end{aligned} \tag{7}$$

Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ be a unital and counital BiHom-bialgebra with a unit 1_H and a counit ε_H. A linear map $S : H \longrightarrow H$ is called an antipode if it commutes with all the maps $\alpha, \beta, \psi, \omega$, and it satisfies the following relation:

$$\beta\psi(S(h_1))\alpha\omega(h_2) = \varepsilon_H(h)1_H = \beta\psi(h_1)\alpha\omega(S(h_2)), \forall h \in H. \tag{8}$$

A BiHom-Hopf algebra is a unital and counital BiHom-bialgebra with an antipode.

We can get some properties of the antipode. The proof is similar to the monoidal BiHom-Hopf algebra case in ([7], Proposition 6.6).

Proposition 4. *Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega, S)$ be a BiHom-Hopf algebra. Then,*

$$\begin{gathered}S(1_H) = 1_H, \varepsilon_H \circ S = \varepsilon_H,\\ S(\beta(a)\alpha(b)) = S(\beta(b))S(\alpha(a)), \forall a, b \in H,\\ \psi\left(S(h)_1\right) \otimes \omega\left(S(h)_2\right) = \omega(S(h_2)) \otimes \psi(S(h_1)), \forall h \in H.\end{gathered} \tag{9}$$

Definition 5 ([7, 10]).

Let $(A, \mu_A, \alpha_A, \beta_A)$ be a BiHom-associative algebra and (M, α_M, β_M) a triple where M is a linear space, and α_M, $\beta_M : M \longrightarrow M$are commuting linear maps.

(i) (M, α_M, β_M) is a left A-module if we have a linear map $A \otimes M \longrightarrow M$, $a \otimes m \mapsto a \cdot m$, such that $\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)$, $\beta_M(a \cdot m) = \beta_A(a) \cdot \beta_M(m)$, and

$$\alpha_A(a) \cdot \left(a' \cdot m\right) = \left(a \cdot a'\right) \cdot \beta_M(m), \forall a, a' \in A, m \in M. \tag{10}$$

If (M, α_M, β_M) and (N, α_N, β_N) are left A-modules (both A-actions denoted by $\cdot$), a morphism of left A-modules $f : M \longrightarrow N$ is a linear map satisfying the conditions $\alpha_N \circ f = f \circ \alpha_M$, $\beta_N \circ f = f \circ \beta_M$ and $f(a \cdot m) = a \cdot f(m)$, for all $a \in A$ and $m \in M$.

If $(A, \mu_A, \alpha_A, \beta_A, 1_A)$ is a unital BiHom-associative algebra and (M, α_M, β_M) is a left A-module, then M is called unital if $1_A \cdot m = \beta_M(m)$, for all $m \in M$.

(ii) (M, α_M, β_M) is a right A-module if we have a linear map $M \otimes A \longrightarrow M$, $m \otimes a \mapsto m \cdot a$, such that $\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a)$, $\beta_M(m \cdot a) = \beta_M(m) \cdot \beta_A(a)$, and

$$\alpha_M(m) \cdot \left(a \cdot a'\right) = (m \cdot a) \cdot \beta_A\left(a'\right), \forall a, a' \in A, m \in M. \tag{11}$$

If $(A, \mu_A, \alpha_A, \beta_A, 1_A)$ is a unital BiHom-associative algebra and (M, α_M, β_M) is a right A-module, then M is called unital if $m \cdot 1_A = \alpha_M(m)$, for all $m \in M$.

(iii) If (M, α_M, β_M) is a left (A, α_A, β_A)-module and a right (B, α_B, β_B)-module, then M is called an (A, B)-bimodule if

$$\alpha_A(a) \cdot \left(m \cdot a'\right) = (a \cdot m) \cdot \beta_B\left(a'\right), \forall a \in A, a' \in B, m \in M. \tag{12}$$

(iv) Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra for which the maps $\alpha_H, \beta_H, \psi_H, \omega_H$ are bijective. A BiHom-associative algebra $(A, \mu_A, \alpha_A, \beta_A)$ is called a left H-module BiHom-algebra if (A, α_A, β_A) is a left (H, α_H, β_H)-module, with action denoted by $H \otimes A \longrightarrow A$, $h \otimes a \mapsto h \cdot a$, such that the following condition is satisfied:

$$h \cdot \left(aa'\right) = \left[\alpha_H^{-1}\omega_H^{-1}(h_1) \cdot a\right]\left[\beta_H^{-1}\psi_H^{-1}(h_2) \cdot a'\right], \forall h \in H, a, a' \in A. \tag{13}$$

3. The Maschke-Type Theorem for the BiHom-Smash Product $A\#H$

In this section, we will give a Maschke-type theorem for the BiHom-smash product $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ over a semisimple BiHom-Hopf algebra $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H, S_H)$.

Definition 6 ([7]). Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra and $(A, \mu_A, \alpha_A, \beta_A)$ a left H-module BiHom-algebra, with the left action $H \otimes A \longrightarrow A$, $h \otimes a \mapsto h \cdot a$ such that all structure maps $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ are bijective. The BiHom-smash product $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ is defined on the vector space $A \otimes H$, and the BiHom-multiplication is given by

$$(a\#h)\left(a'\#h'\right) = a\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}\left(a'\right)\right)\#\psi_H^{-1}(h_2)h', \tag{14}$$

for all $a, a' \in A$, $h, h' \in H$. Note that $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ is a BiHom-associative algebra. Moreover, if A and H are both unital with the units 1_A and 1_H, then $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ also has a unit $1_A\#1_H$.

We assume that the BiHom-smash product in our paper is unital.

Proposition 7. *Let $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ be a BiHom-smash product, then there are two BiHom-algebra isomorphisms $A \cong A\#1_H$ via $a \mapsto a\#1_H$ and $H \cong 1_A\#H$ via $h \mapsto 1_A\#h$. This means $(a\#1_H)(b\#1_H) = ab\#1_H$ and $(1_A\#h)(1_A\#g) = 1_A\#hg$, for all $a, b \in A$ and $h, g \in H$.*

Proof. A straightforward computation left to the reader.

Our next result is the BiHom-analogue of the integral (for a Hom-analogue, see [17] and monoidal Hom-analogue, see [16]).

Definition 8. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H, 1_H, \varepsilon_H, S_H)$ be a BiHom-Hopf algebra. A left integral in H is an element $t \in H$ which is α_H and β_H-invariant (i.e., $\alpha_H(t) = t$, $\beta_H(t) = t$) such that

$$ht = \varepsilon(h)t, \tag{15}$$

for all $h \in H$. A left integral t is normalized if $\varepsilon_H(t) = 1$. Similarly, we can define right integrals in H. We denote the space of left and right integrals in H by $\int_l^H$ and $\int_r^H$. If $\int_l^H = \int_r^H$, then we say H is unimodular. H is semisimple if and only if H possesses a normalized left integral if and only if H possesses a normalized right integral.

Proposition 9. *Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H, 1_H, \varepsilon_H, S_H)$ be a finite-dimensional BiHom-Hopf algebra with $0 \neq t \in \int_r^H$, (A, α_A, β_A) be an (H, α_H, β_H)-module BiHom-algebra, and (M, α_M, β_M), (N, α_N, β_N), be left $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$-modules. If $\lambda : (M, \alpha_M, \beta_M) \longrightarrow (N, \alpha_N, \beta_N)$ is an (A, α_A, β_A)-module map, then*

$$\begin{aligned} \tilde{\lambda} : (M, \alpha_M, \beta_M) &\longrightarrow (N, \alpha_N, \beta_N), \\ m &\mapsto \beta_H\omega_H^{-1}S_H(t_1) \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(m)\right) \end{aligned} \tag{16}$$

is an $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$-module morphism, where $\Delta_H(t) = t_1 \otimes t_2$.

Proof. First, show that $\alpha_N \circ \tilde{\lambda} = \tilde{\lambda} \circ \alpha_M$. For all $m \in M$,

$$\begin{aligned}
\alpha_N \circ \tilde{\lambda}(m) &= \alpha_N\left(\tilde{\lambda}(m)\right) \\
&= \alpha_H \beta_H \omega_H^{-1} S_H(t_1) \cdot \alpha_N \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(m)\right) \\
&= \beta_H \omega_H^{-1} S_H(\alpha_H(t_1)) \cdot \lambda\left(\psi_H^{-1}(\alpha_H(t_2)) \cdot \alpha_M \beta_M^{-2}(m)\right) \\
&= \beta_H \omega_H^{-1} S_H(t_1) \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(\alpha_M(m))\right) \\
&\quad \cdot (by\, \alpha_H(t) = t) \\
&= \tilde{\lambda}(\alpha_M(m)).
\end{aligned} \tag{17}$$

Similarly, from $\beta_H(t) = t$, we get $\beta_N(\tilde{\lambda}(m)) = \tilde{\lambda}(\beta_M(m))$.

Since $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ is finite-dimensional and there is a nonzero integral t, it follows that S_H is bijective. Meanwhile, all structure maps $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ are bijective; thus, for $g \in H$, there exists an element $h \in H$ such that $g = S_H(\alpha_H \beta_H^3 \omega_H \psi_H^2(h))$. For any $m \in M$, we have

$$\begin{aligned}
g \cdot \tilde{\lambda}(m) &= S_H\left(\alpha_H \beta_H^3 \omega_H \psi_H^2(h)\right) \cdot \left[\beta_H \omega_H^{-1} S_H(t_1)\right. \\
&\quad \left. \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(m)\right)\right] \\
&\overset{(7)}{=} \left[S_H\left(\beta_H^3 \omega_H \psi_H^2(h)\right) S_H\left(\beta_H \omega_H^{-1}(t_1)\right)\right] \\
&\quad \cdot \lambda\left(\beta_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&= \left[S_H \beta_H\left(\beta_H^2 \omega_H \psi_H^2(h)\right) S_H \alpha_H\left(\omega_H^{-1}(t_1)\right)\right] \\
&\quad \cdot \lambda\left(\alpha_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) (by\, \alpha_H(t) = t) \\
&\overset{(75)}{=} S_H\left(\beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \omega_H \psi_H^2(h)\right) \cdot \lambda\left(\alpha_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&= S_H\left(\beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \psi_H^2(h_1) \varepsilon_H(h_2)\right) \\
&\quad \cdot \lambda\left(\alpha_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&= S_H\left(\beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \psi_H^2(h_1)\right) \\
&\quad \cdot \lambda\left(\left(\psi_H^{-1}(t_2) 1_H \varepsilon_H(h_2)\right) \cdot \beta_M^{-1}(m)\right) \\
&\overset{(753)}{=} S_H\left(\alpha_H \beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \psi_H^2(h_1)\right) \\
&\quad \cdot \lambda\left(\left(\alpha_H \psi_H^{-1}(t_2)(\beta_H \psi_H(h_{21}) \alpha_H \omega_H S_H(h_{22}))\right) \cdot \beta_M^{-1}(m)\right) \\
&\overset{(7531)}{=} S_H\left(\alpha_H \beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \psi_H^2(h_1)\right) \\
&\quad \cdot \lambda\left[\left(\left(\psi_H^{-1}(t_2) \beta_H \psi_H(h_{21})\right) \alpha_H \beta_H \omega_H S_H(h_{22})\right) \cdot \beta_M^{-1}(m)\right] \\
&\overset{(75312)}{=} S_H\left(\alpha_H \beta_H \omega_H^{-1}(t_1) \alpha_H \beta_H^2 \omega_H^{-1} \psi_H^2(h_{11})\right) \\
&\quad \cdot \lambda\left[\left(\left(\psi_H^{-1}(t_2) \beta_H \psi_H(h_{12})\right) \alpha_H \beta_H \omega_H \psi_H S_H(h_2)\right) \cdot \beta_M^{-1}(m)\right] \\
&= S_H \alpha_H \beta_H \omega_H^{-1}\left(t_1 \beta_H \psi_H^2(h_{11})\right) \\
&\quad \cdot \lambda\left[\left(\psi_H^{-1}\left(t_2 \beta_H \psi_H^2(h_{12})\right) \alpha_H \beta_H \omega_H \psi_H S_H(h_2)\right) \cdot \beta_M^{-1}(m)\right] \\
&= S_H \alpha_H \beta_H \omega_H^{-1}(t_1 \varepsilon_H(h_1)) \\
&\quad \cdot \lambda\left[\left(\psi_H^{-1}(t_2) \alpha_H \beta_H \omega_H \psi_H S_H(h_2)\right) \cdot \beta_M^{-1}(m)\right] \\
&= S_H \alpha_H \beta_H \omega_H^{-1}(t_1) \cdot \lambda\left[\left(\psi_H^{-1}(t_2) \alpha_H \beta_H \omega_H \psi_H^2 S_H(h)\right) \cdot \beta_M^{-1}(m)\right] \\
&= \beta_H \omega_H^{-1}(S_H(t_1)) \cdot \lambda\left[\left(\alpha_H^{-1} \psi_H^{-1}(t_2) \beta_H^{-2}(g)\right) \cdot \beta_M^{-1}(m)\right] \\
&\overset{(753127)}{=} \beta_H \omega_H^{-1}(S_H(t_1)) \cdot \lambda\left[\psi_H^{-1}(t_2) \cdot \left(\beta_H^{-2}(g) \cdot \beta_M^{-2}(m)\right)\right] \\
&= \beta_H \omega_H^{-1}(S_H(t_1)) \cdot \lambda\left[\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(g \cdot m)\right] = \tilde{\lambda}(g \cdot m),
\end{aligned} \tag{18}$$

which implies that $\tilde{\lambda}$ is a left (H, α_H, β_H)-module morphism. Furthermore, we have

$$\begin{aligned}
(1_A \# h)(a \# 1_H) &= \omega_H^{-1}(h_1) \cdot a \# \alpha_H \psi_H^{-1}(h_2), \\
(a \# 1_H)(1_A \# h) &= \alpha_A(a) \# \beta_H(h),
\end{aligned} \tag{19}$$

for all $a \in A$, $h \in H$. Meanwhile,

$$a \# h = \left(1_A \# \alpha_H^{-1} \psi_H^{-1}(h_2)\right)\left(S_H^{-1}\left(\alpha_H^{-1} \beta_H^{-1} \omega_H^{-1}(h_1)\right) \cdot \beta_A^{-2}(a) \# 1_H\right) \tag{20}$$

obtains by the following computation:

$$\begin{aligned}
&\left(1_A \# \alpha_H^{-1} \psi_H^{-1}(h_2)\right)\left(S_H^{-1}\left(\alpha_H^{-1} \beta_H^{-1} \omega_H^{-1}(h_1)\right) \cdot \beta_A^{-2}(a) \# 1_H\right) \\
&\overset{(2)}{=} \left[\alpha_H^{-1} \psi_H^{-1} \omega_H^{-1}(h_{21}) \cdot \left(S_H^{-1} \alpha_H^{-1} \beta_H^{-1} \omega_H^{-1}(h_1) \cdot \beta_A^{-2}(a)\right)\right] \# \psi_H^{-2}(h_{22}) \\
&\overset{(27)}{=} \left[\alpha_H^{-2} \psi_H^{-1} \omega_H^{-1}(h_{21}) S_H^{-1} \alpha_H^{-1} \beta_H^{-1} \omega_H^{-1}(h_1)\right] \cdot \beta_A^{-1}(a) \# \psi_H^{-2}(h_{22}) \\
&\overset{(272)}{=} \left[\alpha_H^{-2} \psi_H^{-1} \omega_H^{-1}(h_{12}) S_H^{-1} \alpha_H^{-1} \beta_H^{-1} \omega_H^{-2}(h_{11})\right] \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(h_2) \\
&= \left[S_H^{-1} \beta_H\left(S_H \alpha_H^{-2} \beta_H^{-1} \psi_H^{-1} \omega_H^{-1}(h_{12})\right) S_H^{-1} \alpha_H\left(\alpha_H^{-2} \beta_H^{-1} \omega_H^{-2}(h_{11})\right)\right] \\
&\quad \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(h_2) \\
&\overset{(2725)}{=} S_H^{-1}\left[\alpha_H^{-2} \omega_H^{-2}(h_{11}) S_H \alpha_H^{-1} \beta_H^{-1} \psi_H^{-1} \omega_H^{-1}(h_{12})\right] \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(h_2) \\
&= S_H^{-1}\left[\beta_H \psi_H\left(\alpha_H^{-2} \beta_H^{-1} \omega_H^{-2} \psi_H^{-1}(h_{11})\right) S_H \alpha_H \omega_H\left(\alpha_H^{-1} \beta_H^{-2} \psi_H^{-1} \omega_H^{-2}(h_{12})\right)\right] \\
&\quad \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(h_2) \\
&\overset{(27253)}{=} S_H^{-1}(1_H) \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(h_2) \varepsilon_H(h_1) \\
&\overset{(272534)}{=} 1_H \cdot \beta_A^{-1}(a) \# \psi_H^{-1}(\varepsilon_H(h_1) h_2) = a \# h,
\end{aligned} \tag{21}$$

for all $a \in A$, $h \in H$. Next, we claim that $\tilde{\lambda}$ is a left (A, α_A, β_A)-module morphism. Indeed, for all $a \in A$, $m \in M$, we get

$$\begin{aligned}
a \cdot \tilde{\lambda}(m) &= a \cdot \left[\beta_H \omega_H^{-1} S_H(t_1) \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(m)\right)\right] \\
&= (a \# 1_H) \cdot \left[\left(1_A \# \beta_H \omega_H^{-1} S_H(t_1)\right) \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-2}(m)\right)\right] \\
&\overset{(7)}{=} \left[\left(\alpha_A^{-1}(a) \# 1_H\right)\left(1_A \# \beta_H \omega_H^{-1} S_H(t_1)\right)\right] \\
&\quad \cdot \lambda\left(\beta_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&\overset{(73)}{=} \left(a \# \beta_H^2 \omega_H^{-1} S_H(t_1)\right) \cdot \lambda\left(\beta_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&= \left(a \# \beta_H \omega_H^{-1} S_H(t_1)\right) \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) (by\, \beta_H(t) = t) \\
&\overset{(734)}{=} \left[\left(1_A \# \alpha_H^{-1} \beta_H \omega_H^{-1} \psi_H^{-1}\left(S_H(t_1)_2\right)\right)\left(S_H^{-1} \alpha_H^{-1} \omega_H^{-2}\left(S_H(t_1)_1\right)\right.\right. \\
&\quad \left.\left. \cdot \beta_A^{-2}(a) \# 1_H\right)\right] \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&\overset{(7346)}{=} \left[\left(1_A \# \alpha_H^{-1} \beta_H \omega_H^{-2} S_H(t_{11})\right)\left(\alpha_H^{-1} \omega_H^{-1} \psi_H^{-1}(t_{12})\right.\right. \\
&\quad \left.\left. \cdot \beta_A^{-2}(a) \# 1_H\right)\right] \cdot \lambda\left(\psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) \\
&= \left[\left(1_A \# \beta_H \omega_H^{-2} S_H(t_{11})\right)\left(\omega_H^{-1} \psi_H^{-1}(t_{12}) \cdot \beta_A^{-2}(a) \# 1_H\right)\right] \\
&\quad \cdot \lambda\left(\alpha_H \psi_H^{-1}(t_2) \cdot \beta_M^{-1}(m)\right) (by\, \alpha_H(t) = t) \\
&\overset{(73462)}{=} \left[\left(1_A \# \beta_H \omega_H^{-1} S_H(t_1)\right)\left(\omega_H^{-1} \psi_H^{-1}(t_{21}) \cdot \beta_A^{-2}(a) \# 1_H\right)\right] \\
&\quad \cdot \lambda\left(\alpha_H \psi_H^{-2}(t_{22}) \cdot \beta_M^{-1}(m)\right)
\end{aligned}$$

$$\begin{aligned}
&\overset{(734627)}{=}\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\left[\left(\omega_H^{-1}\psi_H^{-1}(t_{21})\cdot\beta_A^{-2}(a)\#1_H\right)\right.\\
&\quad\left.\cdot\lambda\left(\alpha_H\beta_H^{-1}\psi_H^{-2}(t_{22})\cdot\beta_M^{-2}(m)\right)\right]\\
&=\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\lambda\left[\left(\omega_H^{-1}\psi_H^{-1}(t_{21})\cdot\beta_A^{-2}(a)\#1_H\right)\right.\\
&\quad\left.\cdot\left(\left(1_A\#\alpha_H\beta_H^{-1}\psi_H^{-2}(t_{22})\right)\cdot\beta_M^{-2}(m)\right)\right]\\
&\overset{(7346277)}{=}\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\lambda\left[\left(\left(\alpha_H^{-1}\omega_H^{-1}\psi_H^{-1}(t_{21})\right.\right.\right.\\
&\quad\left.\left.\left.\cdot\alpha_A^{-1}\beta_A^{-2}(a)\#1_H\right)\left(1_A\#\alpha_H\beta_H^{-1}\psi_H^{-2}(t_{22})\right)\right)\cdot\beta_M^{-1}(m)\right]\\
&\overset{(73462773)}{=}\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\lambda\left[\left(\omega_H^{-1}\psi_H^{-1}(t_{21})\right.\right.\\
&\quad\left.\left.\cdot\beta_A^{-2}(a)\#\alpha_H\psi_H^{-2}(t_{22})\right)\cdot\beta_M^{-1}(m)\right]\\
&=\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\lambda\left[\left(\omega_H^{-1}\left(\psi_H^{-1}(t_{21})\right)\right.\right.\\
&\quad\left.\left.\cdot\beta_A^{-2}(a)\#\alpha_H\psi_H^{-1}\left(\psi_H^{-1}(t_{22})\right)\right)\cdot\beta_M^{-1}(m)\right]\\
&\overset{(734627732)}{=}\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\\
&\quad\cdot\lambda\left[\left(\left(1_A\#\psi_H^{-1}(t_2)\right)\left(\beta_A^{-2}(a)\#1_H\right)\cdot\beta_M^{-1}(m)\right]\right.\\
&\overset{(7346277327)}{=}\left(1_A\#\alpha_H\beta_H\omega_H^{-1}S_H(t_1)\right)\\
&\quad\cdot\lambda\left[\left(1_A\#\alpha_H\psi_H^{-1}(t_2)\cdot\left(\left(\beta_A^{-2}(a)\#1_H\right)\cdot\beta_M^{-2}(m)\right)\right]\right.\\
&=\left(1_A\#\beta_H\omega_H^{-1}S_H(t_1)\right)\cdot\lambda\left[\left(1_A\#\psi_H^{-1}(t_2)\right)\right.\\
&\quad\left.\cdot\left(\beta_A^{-2}(a)\cdot\beta_M^{-2}(m)\right)\right]\ (\text{by }\alpha_H(t)=t)\\
&=\beta_H\omega_H^{-1}S_H(t_1)\cdot\lambda\left(\psi_H^{-1}(t_2)\cdot\beta_M^{-2}(a\cdot m)\right)=\tilde{\lambda}(a\cdot m).
\end{aligned}\tag{22}$$

Thus, we get that $\tilde{\lambda}$ is a left $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-module morphism.

Proposition 10. *Let $(H,\mu_H,\Delta_H,\alpha_H,\beta_H,\psi_H,\omega_H,1_H,\varepsilon_H,S_H)$ be a finite-dimensional semisimple BiHom-Hopf algebra and (A,α_A,β_A) be a left (H,α_H,β_H)-module BiHom-algebra. Let (M,α_M,β_M) be a left $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-module and (N,α_N,β_N) an $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-submodule of (M,α_M,β_M). If (N,α_N,β_N) is a direct summand of (M,α_M,β_M) as (A,α_A,β_A)-modules, then (N,α_N,β_N) is also a direct summand of (M,α_M,β_M) as $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-modules.*

Proof. Since $(H,\mu_H,\Delta_H,\alpha_H,\beta_H,\psi_H,\omega_H,1_H,\varepsilon_H,S_H)$ is a finite-dimensional semisimple BiHom-Hopf algebra, there exists a normalized right integral t in H. Let $\lambda:(M,\alpha_M,\beta_M)\longrightarrow(N,\alpha_N,\beta_N)$ be a canonical projection as (A,α_A,β_A)-modules. Define

$$\begin{aligned}
\tilde{\lambda}:(M,\alpha_M,\beta_M)&\longrightarrow(N,\alpha_N,\beta_N),\\
m&\mapsto\beta_H\omega_H^{-1}S_H(t_1)\cdot\lambda\left(\psi_H^{-1}(t_2)\cdot\beta_M^{-2}(m)\right).
\end{aligned}\tag{23}$$

Thus, by Proposition 9, we obtain that $\tilde{\lambda}$ is a left $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-module morphism. Now, we show that $\tilde{\lambda}$ is also a projection. By the projrctivity of λ, we prove for all $n\in N$ that

$$\begin{aligned}
\tilde{\lambda}(n)&=\beta_H\omega_H^{-1}S_H(t_1)\cdot\lambda\left(\psi_H^{-1}(t_2)\cdot\beta_N^{-2}(n)\right)\\
&=\beta_H\omega_H^{-1}S_H(t_1)\cdot\left(\psi_H^{-1}(t_2)\cdot\beta_N^{-2}(n)\right)\\
&\overset{(7)}{=}\left(\alpha_H^{-1}\beta_H\omega_H^{-1}S_H(t_1)\psi_H^{-1}(t_2)\right)\cdot\beta_N^{-1}(n)\\
&=\left(\beta_H\omega_H^{-1}S_H(t_1)\alpha_H\psi_H^{-1}(t_2)\right)\cdot\beta_N^{-1}(n)\ (\text{by }\alpha_H(t)=t)\\
&=\omega_H^{-1}\psi_H^{-1}\left(\beta_H\psi_HS_H(t_1)\alpha_H\omega_H(t_2)\right)\cdot\beta_N^{-1}(n)\\
&\overset{(73)}{=}\varepsilon_H(t)1_H\cdot\beta_N^{-1}(n)=1_H\cdot\beta_N^{-1}(n)=n.
\end{aligned}\tag{24}$$

It follows (N,α_N,β_N) which is a direct summand of (M,α_M,β_M) as left $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-modules.

By the above discussions, we obtain the Maschke-type theorem for BiHom-smash product, which generalizes Theorem 14 in [16].

Theorem 11. *Let $(H,\mu_H,\Delta_H,\alpha_H,\beta_H,\psi_H,\omega_H,1_H,\varepsilon_H,S_H)$ be a finite-dimensional semisimple BiHom-Hopf algebra and (A,α_A,β_A) be a left (H,α_H,β_H)-module BiHom-algebra. If (A,α_A,β_A) is semisimple, then so is the BiHom-smash product $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$.*

4. The Associated Morita Context

The main aim of this section is to construct an associated Morita context between the BiHom-smash product $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$ and the BiHom-subalgebra $(A^H,\alpha_A|_{A^H},\beta_A|_{A^H})$. Note that the monoidal Hom-analogue of the Morita context has been studied in [16].

Let (A,α_A,β_A) be a left (H,α_H,β_H)-module BiHom-algebra and

$$A^H=\{a\in A\mid h\cdot a=\varepsilon(h)\beta_A(a),\forall h\in H\}\tag{25}$$

be the (H,α_H,β_H)-invariants of (A,α_A,β_A). It is easy to get $1_A\in A^H$. If $a,a'\in A^H$, then both $\alpha_A(a),\beta_A(a)$ and aa' are in A^H. It follows $(A^H,\alpha_A|_{A^H},\beta_A|_{A^H})$ which is a BiHom-subalgebra of (A,α_A,β_A). For all $a,a'\in A^H,h\in H$, we only check

$$\begin{aligned}
h\cdot\left(aa'\right)&\overset{(10)}{=}\left[\alpha_H^{-1}\left(\omega_H^{-1}(h_1)\right)\cdot a\right]\left[\beta_H^{-1}\left(\psi_H^{-1}(h_2)\right)\cdot a'\right]\\
&=\left(\varepsilon_H(h_1)\beta_A(a)\right)\left(\varepsilon_H(h_2)\beta_A\left(a'\right)\right)\\
&=\varepsilon_H(h)\beta_A\left(aa'\right).
\end{aligned}\tag{26}$$

Let t be a normalized left integral of (H, α_H, β_H); then for all $h \in H$, $a \in A$, we have

$$h \cdot (t \cdot a) \overset{(7)}{=} (\alpha_H^{-1}(h)t) \cdot \beta_A(a) = \varepsilon_H(h)t \cdot \beta_A(a) = \varepsilon_H(h)\beta_H(t) \cdot \beta_A(a) = \varepsilon_H(h)\beta_H(t \cdot a). \quad (27)$$

This means $t \cdot a \in A^H$. Conversely, if $a \in A^H$, then $\beta_A^{-1}(a) \in A^H$ and $t \cdot \beta_A^{-1}(a) = \varepsilon_H(t)a = a$. So, $A^H \subset t \cdot A$. In summary, if $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H, 1_H, \varepsilon_H, S_H)$ is a BiHom-Hopf algebra with a normalized left integral t, then $t \cdot A = A^H$.

Lemma 12. *Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-Hopf algebra with a bijective antipode S_H and (A, α_A, β_A) be a left (H, α_H, β_H)-module BiHom-algebra. Then, (A, α_A, β_A) is a left and right $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$-module via the following module structure maps:*

(i) $(a\#h) \longrightarrow b = a\beta_A^{-1}(h \cdot b)$,

(ii) $b \longleftarrow (a\#h) = S_H^{-1}\alpha_H^{-1}\beta_H(h) \cdot \beta_A^{-1}(ba)$,

for all $a, b \in A$, $h \in H$.

Proof. We prove the conditions (10) and (11) hold and leave the others to the reader. For (10),

$$\begin{aligned}
&[(a\#h)(b\#k)] \longrightarrow \beta_A(c) \\
&\overset{(1)}{=} \left[a\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)\right)\#\psi_H^{-1}(h_2)k\right] \longrightarrow \beta_A(c) \\
&= \left(a\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)\right)\right)\beta_A^{-1}\left(\left(\psi_H^{-1}(h_2)k\right) \cdot \beta_A(c)\right) \\
&= \left(a\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)\right)\right)\left(\left(\beta_H^{-1}\psi_H^{-1}(h_2)\beta_H^{-1}(k)\right) \cdot c\right) \\
&\overset{(17)}{=} \alpha_A(a)\left[\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)\right) \times \left(\left(\beta_H^{-2}\psi_H^{-1}(h_2)\beta_H^{-2}(k)\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&\overset{(177)}{=} \alpha_A(a)\left[\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)\right)\left(\alpha_H\beta_H^{-2}\psi_H^{-1}(h_2) \cdot \left(\beta_H^{-2}(k) \cdot \beta_A^{-2}(c)\right)\right)\right] \\
&= \alpha_A(a)\left[\left(\alpha_H^{-1}\omega_H^{-1}\left(\alpha_H\beta_H^{-1}(h_1)\right) \cdot \beta_A^{-1}(b)\right)\left(\beta_H^{-1}\psi_H^{-1}\left(\alpha_H\beta_H^{-1}(h_2)\right) \cdot \left(\beta_H^{-2}(k) \cdot \beta_A^{-2}(c)\right)\right)\right] \\
&\overset{(17710)}{=} \alpha_A(a)\left[\alpha_H\beta_H^{-1}(h) \cdot \left(\beta_A^{-1}(b)\left(\beta_H^{-2}(k) \cdot \beta_A^{-2}(c)\right)\right)\right] \\
&= \alpha_A(a)\beta_A^{-1}\left[\alpha_H(h) \cdot \left(b\beta_A^{-1}(k \cdot c)\right)\right] \\
&= (\alpha_A(a)\#\alpha_H(h)) \longrightarrow \left(b\beta_A^{-1}(k \cdot c)\right) \\
&= (\alpha_A(a)\#\alpha_H(h)) \longrightarrow ((b\#k) \longrightarrow c).
\end{aligned} \quad (28)$$

For (11),

$$\begin{aligned}
&\alpha_A(a) \longleftarrow ((b\#h)(c\#k)) \\
&\overset{(1)}{=} \alpha_A(a) \longleftarrow \left[b\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(c)\right)\#\psi_H^{-1}(h_2)k\right] \\
&= S_H^{-1}\alpha_H^{-1}\beta_H\left(\psi_H^{-1}(h_2)k\right) \cdot \beta_A^{-1}\left[\alpha_A(a)\left(b\left(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(c)\right)\right)\right] \\
&\overset{(17)}{=} \alpha_H^{-1}\beta_H S_H^{-1}\left(\beta_H\left(\beta_H^{-1}\psi_H^{-1}(h_2)\right)\alpha_H\left(\alpha_H^{-1}(k)\right)\right) \cdot \beta_A^{-1}\left((ab)\left(\omega_H^{-1}(h_1) \cdot c\right)\right) \\
&\overset{(175)}{=} \left[\alpha_H^{-1}\beta_H S_H^{-1}\beta_H\left(\alpha_H^{-1}(k)\right)\alpha_H^{-1}\beta_H S_H^{-1}\alpha_H\left(\beta_H^{-1}\psi_H^{-1}(h_2)\right)\right] \cdot \beta_A^{-1}\left((ab)\left(\omega_H^{-1}(h_1) \cdot c\right)\right) \\
&= \left[\alpha_H^{-2}\beta_H^{2}S_H^{-1}(k)S_H^{-1}\psi_H^{-1}(h_2)\right] \cdot \beta_A^{-1}\left((ab)\left(\omega_H^{-1}(h_1) \cdot c\right)\right) \\
&\overset{(1757)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[S_H^{-1}\psi_H^{-1}(h_2) \cdot \beta_A^{-2}\left((ab)\left(\omega_H^{-1}(h_1) \cdot c\right)\right)\right] \\
&\overset{(175710)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\omega_H^{-1}\left(\psi_H^{-1}S_H^{-1}(h_2)_1\right) \cdot \beta_A^{-2}(ab)\right) \times \left(\beta_H^{-1}\psi_H^{-1}\left(\psi_H^{-1}S_H^{-1}(h_2)_2\right) \cdot \left(\beta_H^{-2}\omega_H^{-1}(h_1) \cdot \beta_A^{-2}(c)\right)\right)\right] \\
&\overset{(1757106)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-2}S_H^{-1}(h_{22}) \cdot \beta_A^{-2}(ab)\right) \times \left(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}S_H^{-1}(h_{21}) \cdot \left(\beta_H^{-2}\omega_H^{-1}(h_1) \cdot \beta_A^{-2}(c)\right)\right)\right] \\
&\overset{(17571067)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-2}S_H^{-1}(h_{22}) \cdot \beta_A^{-2}(ab)\right) \times \left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}S_H^{-1}(h_{21})\beta_H^{-2}\omega_H^{-1}(h_1)\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&\overset{(175710672)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-1}S_H^{-1}(h_2) \cdot \beta_A^{-2}(ab)\right) \times \left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}S_H^{-1}(h_{12})\beta_H^{-2}\omega_H^{-2}(h_{11})\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&= \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-1}S_H^{-1}(h_2)\right) \cdot \beta_A^{-2}(ab)\right) \times \left(\left(S_H^{-1}\beta_H\left(\alpha_H^{-1}\beta_H^{-2}\omega_H^{-1}\psi_H^{-1}(h_{12})\right)S_H^{-1}\alpha_H \times \left(\alpha_H^{-1}\beta_H^{-2}\omega_H^{-2}S_H(h_{11})\right)\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&\overset{(1757106725)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-1}S_H^{-1}(h_2)\right) \cdot \beta_A^{-2}(ab)\right) \times \left(S_H^{-1}\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-2}S_H(h_{11})\beta_H^{-2}\omega_H^{-1}\psi_H^{-1}(h_{12})\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&= \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-1}S_H^{-1}(h_2)\right) \cdot \beta_A^{-2}(ab)\right) \times \left(S_H^{-1}\left(\beta_H\psi_H\left(\alpha_H^{-1}\beta_H^{-2}\omega_H^{-2}\psi_H^{-1}S_H(h_{11})\right)\alpha_H\omega_H \times \left(\alpha_H^{-1}\beta_H^{-2}\omega_H^{-2}\psi_H^{-1}(h_{12})\right)\right) \cdot \beta_A^{-1}(c)\right)\right] \\
&\overset{(17571067253)}{=} \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left[\left(\alpha_H^{-1}\psi_H^{-1}S_H^{-1}(h_2) \cdot \beta_A^{-2}(ab)\right)\varepsilon_H(h_1) \times \left(1_H \cdot \beta_A^{-1}(c)\right)\right] \\
&= \alpha_H^{-1}\beta_H^{2}S_H^{-1}(k) \cdot \left(\left(\alpha_H^{-1}S_H^{-1}(h) \cdot \beta_A^{-2}(ab)\right)c\right) \\
&= S_H^{-1}\alpha_H^{-1}\beta_H(\beta_H(k)) \cdot \beta_A^{-1}\left(\left(\alpha_H^{-1}S_H^{-1}\beta_H(h) \cdot \beta_A^{-1}(ab)\right)\beta_A(c)\right) \\
&= \left(S_H^{-1}\alpha_H^{-1}\beta_H(h) \cdot \beta_A^{-1}(ab)\right) \longleftarrow (\beta_A(c)\#\beta_H(k)) \\
&= (a \longleftarrow (b\#h)) \longleftarrow (\beta_A(c)\#\beta_H(k)),
\end{aligned} \quad (29)$$

for all $a, b, c \in A$ and $h, k \in H$.

Lemma 13. *Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-Hopf algebra with a bijective antipode S_H and (A, α_A, β_A) be a left (H, α_H, β_H)-module BiHom-algebra. From Lemma 12, (A, α_A, β_A) is a left and right $(A\#H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$-module. Meanwhile, it is also a left and right $(A^H, \alpha_A|_{A^H}, \beta_A|_{A^H})$*

-module with the BiHom-multiplication. Thus, there are two bimodules $({}_{A^H}A_{A\#H}, \alpha_A, \beta_A)$ and $({}_{A\#H}A_{A^H}, \alpha_A, \beta_A)$.

Proof. By the BiHom-associativity, it is easy to get (A, α_A, β_A) which is a left and right $(A^H, \alpha_A|_{A^H}, \beta_A|_{A^H})$-module. Now, we check the condition (12) hold for $({}_{A^H}A_{A\#H}, \alpha_A, \beta_A)$ and $({}_{A\#H}A_{A^H}, \alpha_A, \beta_A)$ as follows:

$$\begin{aligned}
(ab) &\longleftarrow (\beta_A(c)\#\beta_H(h)) \\
&= S_H^{-1}\alpha_H^{-1}\beta_H^2(h)\cdot\beta_A^{-1}((ab)\beta_A(c)) \\
&\overset{(7)}{=} S_H^{-1}\alpha_H^{-1}\beta_H^2(h)\cdot\beta_A^{-1}(\alpha_A(a)(bc)) \\
&= S_H^{-1}\alpha_H^{-1}\beta_H^2(h)\cdot\left(\alpha_A\beta_A^{-1}(a)\beta_A^{-1}(bc)\right) \\
&\overset{(710)}{=} \left[\alpha_H^{-1}\omega_H^{-1}\left(S_H^{-1}\alpha_H^{-1}\beta_H^2(h)_1\right)\cdot\alpha_A\beta_A^{-1}(a)\right] \\
&\quad\times\left[\beta_H^{-1}\psi_H^{-1}\left(S_H^{-1}\alpha_H^{-1}\beta_H^2(h)_2\right)\cdot\beta_A^{-1}(bc)\right] \\
&\overset{(7106)}{=} \left[\alpha_H^{-2}\beta_H^{-2}\psi_H^{-1}S_H^{-1}(h_2)\cdot\alpha_A\beta_A^{-1}(a)\right] \\
&\quad\times\left[\alpha_H^{-1}\beta_H\omega_H^{-1}S_H^{-1}(h_1)\cdot\beta_A^{-1}(bc)\right] \\
&= \varepsilon_H(h_2)\alpha_A(a)\left[\alpha_H^{-1}\beta_H\omega_H^{-1}S_H^{-1}(h_1)\cdot\beta_A^{-1}(bc)\right] \\
&= \alpha_A(a)\left[S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(bc)\right] = \alpha_A(a)(b\longleftarrow(c\#h)),
\end{aligned}$$

$$\begin{aligned}
(\alpha_A(c)\#\alpha_H(h)) &\longrightarrow (ba) \\
&= \alpha_A(c)\beta_A^{-1}(\alpha_H(h)\cdot(ba)) \\
&\overset{(10)}{=} \alpha_A(c)\beta_A^{-1}\left((\omega_H^{-1}(h_1)\cdot b)\left(\alpha_H\beta_H^{-1}\psi_H^{-1}(h_2)\cdot a\right)\right) \\
&= \alpha_A(c)\beta_A^{-1}\left((\omega_H^{-1}(h_1)\cdot b)\varepsilon_H(h_2)\beta_A(a)\right) \\
&= \alpha_A(c)\left(\beta_A^{-1}(h\cdot b)a\right) \\
&\overset{(107)}{=} \left(c\beta_A^{-1}(h\cdot b)\right)\beta_A(a) = ((c\#h)\longrightarrow b)\beta_A(a),
\end{aligned} \tag{30}$$

for all $a\in A^H, b, c\in A$ and $h\in H$.

Let t be a left integral in $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$. If $S_H(t) = t$, then t is also a right integral in $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$. For all $t\in H$,

$$\begin{aligned}
th &= S_H(t)S_H\left(S_H^{-1}(h)\right) \\
&= S_H\beta_H(t)S_H\alpha_H\left(\alpha_H^{-1}S_H^{-1}(h)\right) \\
&\overset{(5)}{=} S_H\beta_H\left(\alpha_H^{-1}S_H^{-1}(h)\right)S_H\alpha_H(t) \\
&= \alpha_H^{-1}\beta_H(h)S_H(t) = \alpha_H^{-1}\beta_H(h)t = \varepsilon_H(h)t.
\end{aligned} \tag{31}$$

For any $a\in A, h\in H$,

$$\begin{aligned}
&(1_A\#t)(a\#h) \\
&\overset{(4)}{=} (1_A\#t)\left[\left(1_A\#\alpha_H^{-1}\psi_H^{-1}(h_2)\right)\left(S_H^{-1}\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_A^{-2}(a)\#1_H\right)\right] \\
&\overset{(47)}{=} \left[(1_A\#t)\left(1_A\#\alpha_H^{-1}\psi_H^{-1}(h_2)\right)\right]\left(S_H^{-1}\alpha_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_A^{-1}(a)\#1_H\right) \\
&= \left(1_A\#t\alpha_H^{-1}\psi_H^{-1}(h_2)\right)\left(S_H^{-1}\alpha_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_A^{-1}(a)\#1_H\right) \\
&= \left(1_A\#t\varepsilon_H(h_2)\right)\left(S_H^{-1}\alpha_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_A^{-1}(a)\#1_H\right) \\
&= (1_A\#t)\left(S_H^{-1}\alpha_H^{-1}(h)\cdot\beta_A^{-1}(a)\#1_H\right).
\end{aligned} \tag{32}$$

So, we obtain

$$(1_A\#t)(a\#h) = (1_A\#t)\left(S_H^{-1}\alpha_H^{-1}(h)\cdot\beta_A^{-1}(a)\#1_H\right). \tag{33}$$

Now, with the above preparations, we can construct the associated Morita context between the BiHom-smash product $A\#H$ and the BiHom-subalgebra A^H.

Theorem 14. *Let $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-Hopf algebra with a bijective antipode S_H, t be a left integral in H satisfying $S_H(t) = t$, and (A, α_A, β_A) be a left (H, α_H, β_H)-module BiHom-algebra. Then, $[A^H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, A\#H]$ forms an associated Morita context with the maps*

$$\begin{aligned}
&[-,-]\colon (A\otimes_{A^H}A, \alpha_A\otimes\alpha_A, \beta_A\otimes\beta_A)\longrightarrow(A\#H, \alpha_A\otimes\alpha_H, \beta_A\otimes\beta_H), \\
&\qquad via\ [a,b] = (a\#t)(b\#1_H), \\
&(-,-)\colon (A\otimes_{A\#H}A, \alpha_A\otimes\alpha_A, \beta_A\otimes\beta_A)\longrightarrow\left(A^H, \alpha_A|_{A^H}, \beta_A|_{A^H}\right), \\
&\qquad via\ (a,b) = t\cdot\beta_A^{-1}(ab),
\end{aligned} \tag{34}$$

for all $a, b\in A$.

Proof. We first prove $[-,-]$ is both an $(A\#H, \alpha_A\otimes\alpha_H, \beta_A\otimes\beta_H)$-module map and middle $(A^H, \alpha_A|_{A^H}, \beta_A|_{A^H})$-linear map, which means $[-,-]$ satisfies the following three conditions:

$$\begin{aligned}
(\alpha_A(a)\#\alpha_H(h))[b,c] &= [(a\#h)\longrightarrow b, \beta_A(c)], \\
[a,b](\beta_A(c)\#\beta_H(h)) &= [\alpha_A(a), b\longleftarrow(c\#h)], \\
\left[aa', \beta_A(b)\right] &= \left[\alpha_A(a), a'b\right],
\end{aligned} \tag{35}$$

for all $a, b, c\in A, h\in H$ and $a'\in A^H$.

To prove these conditions, we compute

$$\begin{aligned}
&(\alpha_A(a)\#\alpha_H(h))[b,c] \\
&= (\alpha_A(a)\#\alpha_H(h))[(b\#t)(c\#1_H)] \\
&\overset{(1)}{=} [(a\#h)(b\#t)](\beta_A(c)\#1_H) \\
&\overset{(13)}{=} \left[\left(\left(\alpha_A^{-1}(a)\#1_H\right)\left(1_A\#\beta_H^{-1}(h)\right)\right)\left(\left(\alpha_A^{-1}(b)\#1_H\right)(1_A\#t)\right)\right] \\
&\quad\times(\beta_A(c)\#1_H) \\
&\overset{(131)}{=} \left[(a\#1_H)\left(\left(1_A\#\beta_H^{-1}(h)\right)\left(\left(\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)(1_A\#t)\right)\right)\right] \\
&\quad\times(\beta_A(c)\#1_H) \\
&\overset{(1311)}{=} \left[(a\#1_H)\left(\left(\left(1_A\#\alpha_H^{-1}\beta_H^{-1}(h)\right)\left(\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)\right)(1_A\#t)\right)\right] \\
&\quad\times(\beta_A(c)\#1_H) \\
&\overset{(13112)}{=} \left[(a\#1_H)\left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\alpha_A^{-1}\beta_A^{-1}(b)\#\beta_H^{-1}\psi_H^{-1}(h_2)\right)\right.\right. \\
&\quad\left.\left.\times(1_A\#t)\right)\right](\beta_A(c)\#1_H) \\
&\overset{(131123)}{=} \left[(a\#1_H)\left(\left(\left(\alpha_H^{-2}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\alpha_A^{-2}\beta_A^{-1}(b)\#1_H\right)\right.\right.\right. \\
&\quad\left.\left.\left.\times\left(1_A\#\beta_H^{-2}\psi_H^{-1}(h_2)\right)\right)(1_A\#t)\right)\right](\beta_A(c)\#1_H)
\end{aligned}$$

$$\overset{(1311231)}{=} \left[(a\#1_H)\left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)\right.\right.$$
$$\left.\left.\times\left(\left(1_A\#\beta_H^{-2}\psi_H^{-1}(h_2)\right)(1_A\#t)\right)\right)\right](\beta_A(c)\#1_H)$$
$$=\left[(a\#1_H)\left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)\times\left(1_A\#\beta_H^{-2}\psi_H^{-1}(h_2)t\right)\right)\right](\beta_A(c)\#1_H)$$
$$=\left[(a\#1_H)\left(\left(\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)\times(1_A\#\varepsilon_H(h_2)t)\right)\right](\beta_A(c)\#1_H)$$
$$=\left[(a\#1_H)\left(\left(\alpha_H^{-1}\beta_H^{-1}(h)\cdot\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)(1_A\#t)\right)\right]\times(\beta_A(c)\#1_H)$$
$$\overset{(13112313)}{=}\left[(a\#1_H)\left(\beta_H^{-1}(h)\cdot\beta_A^{-1}(b)\#t\right)\right](\beta_A(c)\#1_H)$$
$$=\left(a\left(1_H\cdot\beta_A^{-1}\left(\beta_A^{-1}(h\cdot b)\right)\#t\right)\right)(\beta_A(c)\#1_H)$$
$$=\left(a\beta_A^{-1}(h\cdot b)\#t\right)(\beta_A(c)\#1_H)=\left[a\beta_A^{-1}(h\cdot b),\beta_A(c)\right]$$
$$=[(a\#h)\longrightarrow b,\beta_A(c)],$$

$$[a,b](\beta_A(c)\#\beta_H(h))$$
$$=[(a\#t)(b\#1_H)](\beta_A(c)\#\beta_H(h))$$
$$\overset{(3)}{=}\left[\left(\left(\alpha_A^{-1}(a)\#1_H\right)(1_A\#t)\right)(b\#1_H)\right]\left(\left(\alpha_A^{-1}\beta_A(c)\#1_H\right)(1_A\#h)\right)$$
$$\overset{(31)}{=}\left[(a\#1_H)\left((1_A\#t)\left(\beta_A^{-1}(b)\#1_H\right)\right)\right]\left(\left(\alpha_A^{-1}\beta_A(c)\#1_H\right)(1_A\#h)\right)$$
$$\overset{(311)}{=}(\alpha_A(a)\#1_H)\left[\left((1_A\#t)\left(\beta_A^{-1}(b)\#1_H\right)\right)\times\left(\left(\alpha_A^{-1}(c)\#1_H\right)\left(1_A\#\beta_H^{-1}(h)\right)\right)\right]$$
$$\overset{(3111)}{=}(\alpha_A(a)\#1_H)\left[(1_A\#t)\left(\left(\beta_A^{-1}(b)\#1_H\right)\times\left(\left(\alpha_A^{-1}\beta_A^{-1}(c)\#1_H\right)\left(1_A\#\beta_H^{-2}(h)\right)\right)\right)\right]$$
$$\overset{(31111)}{=}(\alpha_A(a)\#1_H)\left[(1_A\#t)\left(\left(\left(\alpha_A^{-1}\beta_A^{-1}(b)\#1_H\right)\times\left(\alpha_A^{-1}\beta_A^{-1}(c)\#1_H\right)\right)\left(1_A\#\beta_H^{-1}(h)\right)\right)\right]$$
$$=(\alpha_A(a)\#1_H)\left[(1_A\#t)\left(\left(\alpha_A^{-1}\beta_A^{-1}(bc)\#1_H\right)\left(1_A\#\beta_H^{-1}(h)\right)\right)\right]$$
$$\overset{(311113)}{=}(\alpha_A(a)\#1_H)\left[(1_A\#t)\left(\beta_A^{-1}(bc)\#h\right)\right]$$
$$\overset{(3111131)}{=}(\alpha_A(a)\#1_H)\left[(1_A\#t)\left(S_H^{-1}\alpha_H^{-1}(h)\cdot\beta_A^{-2}(bc)\#1_H\right)\right]$$
$$\overset{(31111311)}{=}[(a\#1_H)(1_A\#t)]\left(S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(bc)\#1_H\right)$$
$$=(\alpha_A(a)\#t)\left(S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(bc)\#1_H\right)$$
$$=\left[\alpha_A(a),S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(bc)\right]=[\alpha_A(a),b\longleftarrow(c\#h)]$$

$$\left[\alpha_A(a),a'b\right]$$
$$=(\alpha_A(a)\#t)\left(a'b\#1_H\right)$$
$$=(\alpha_A(a)\#t)\left(\left(a'\#1_H\right)(b\#1_H)\right)$$
$$\overset{(1)}{=}\left((a\#t)\left(a'\#1_H\right)\right)(\beta_A(b)\#1_H)$$
$$=\left(a\left(\beta_H^{-1}\omega_H^{-1}(t_1)\cdot\beta_A^{-1}\left(a'\right)\right)\#\psi_H^{-1}(t_2)1_H\right)(\beta_A(b)\#1_H)$$
$$=\left(a\varepsilon_H(t_1)a'\#\psi_H^{-1}(t_2)1_H\right)(\beta_A(b)\#1_H)$$
$$=\left(aa'\#t1_H\right)(\beta_A(b)\#1_H)$$
$$=\left(aa'\#t\right)(\beta_A(b)\#1_H)=\left[aa',\beta_A(b)\right]. \tag{36}$$

Next, since $A^H=t\cdot A$, thus the map $(-,-)$ is well defined. We now prove that $(-,-)$ is both an $(A^H,\alpha_A|_{A^H},\beta_A|_{A^H})$-module map and middle $(A\#H,\alpha_A\otimes\alpha_H,\beta_A\otimes\beta_H)$-linear. For any $a,b\in A$, $a'\in A^H$, $h\in H$, we compute

$$\left(a'a,\beta_A(b)\right)=t\cdot\beta_A^{-1}\left(\left(a'a\right)\beta_A(b)\right)$$
$$\overset{(1)}{=}t\cdot\beta_A^{-1}\left(\alpha_A\left(a'\right)(ab)\right)$$
$$=t\cdot\left(\alpha_A\beta_A^{-1}\left(a'\right)\beta_A^{-1}(ab)\right)$$
$$\overset{(110)}{=}\left[\alpha_H^{-1}\omega_H^{-1}(t_1)\cdot\alpha_A\beta_A^{-1}\left(a'\right)\right]\cdot\left[\beta_H^{-1}\psi_H^{-1}(t_2)\cdot\beta_A^{-1}(ab)\right]$$
$$=\varepsilon_H(t_1)\alpha_A\left(a'\right)\left(\beta_H^{-1}\psi_H^{-1}(t_2)\cdot\beta_A^{-1}(ab)\right)$$
$$=\alpha_A\left(a'\right)\left(\beta_H^{-1}(t)\cdot\beta_A^{-1}(ab)\right)$$
$$=\alpha_A\left(a'\right)\left(t\cdot\beta_A^{-1}(ab)\right)=\alpha_A\left(a'\right)(a,b),$$

$$\left(\alpha_A(a),ba'\right)=t\cdot\beta_A^{-1}\left(\alpha_A(a)\left(ba'\right)\right)$$
$$\overset{(1)}{=}t\cdot\beta_A^{-1}\left((ab)\beta_A\left(a'\right)\right)$$
$$=t\cdot\left(\beta_A^{-1}(ab)a'\right)$$
$$\overset{(110)}{=}\left[\alpha_H^{-1}\omega_H^{-1}(t_1)\cdot\beta_A^{-1}(ab)\right]\left[\beta_H^{-1}\psi_H^{-1}(t_2)\cdot a'\right]$$
$$=\left[\alpha_H^{-1}\omega_H^{-1}(t_1)\cdot\beta_A^{-1}(ab)\right]\varepsilon_H(t_2)\beta_A\left(a'\right)$$
$$=\left(\alpha_H^{-1}(t)\cdot\beta_A^{-1}(ab)\right)\beta_A\left(a'\right)$$
$$=\left(t\cdot\beta_A^{-1}(ab)\right)\beta_A\left(a'\right)=(a,b)\beta_A\left(a'\right),$$

$$(a\longleftarrow(c\#h),\beta_A(b))=\left(S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(ac),\beta_A(b)\right)$$
$$=t\cdot\beta_A^{-1}\left[\left(S_H^{-1}\alpha_H^{-1}\beta_H(h)\cdot\beta_A^{-1}(ac)\right)\beta_A(b)\right]$$
$$=t\cdot\left[\left(S_H^{-1}\alpha_H^{-1}(h)\cdot\beta_A^{-2}(ac)\right)b\right]$$
$$=t\cdot\left[\left(S_H^{-1}\alpha_H^{-1}\psi_H^{-1}(h_2)\cdot\beta_A^{-2}(ac)\right)(\varepsilon_H(h_1)b)\right]$$
$$\overset{(3)}{=}t\cdot\left[\left(S_H^{-1}\alpha_H^{-1}\psi_H^{-1}(h_2)\cdot\beta_A^{-2}(ac)\right)\cdot\left(\left(S_H^{-1}\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(h_{12})\beta_H^{-2}\omega_H^{-2}(h_{11})\right)\cdot\beta_A^{-1}(b)\right)\right]$$

$$\begin{aligned}
&\overset{(32)}{=} t\cdot\left[\left(S_H^{-1}\alpha_H^{-1}\psi_H^{-2}(h_{22})\cdot\beta_A^{-2}(ac)\right)\right.\\
&\quad\cdot\left(\left(S_H^{-1}\alpha_H^{-1}\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(h_{21})\beta_H^{-2}\omega_H^{-1}(h_1)\right)\right.\\
&\quad\left.\left.\cdot\beta_A^{-1}(b)\right)\right]\\
&\overset{(327)}{=} t\cdot\left[\left(S_H^{-1}\alpha_H^{-1}\psi_H^{-2}(h_{22})\cdot\beta_A^{-2}(ac)\right)\right.\\
&\quad\cdot\left(S_H^{-1}\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(h_{21})\right.\\
&\quad\left.\left.\cdot\left(\beta_H^{-2}\omega_H^{-1}(h_1)\cdot\beta_A^{-2}(b)\right)\right)\right]\\
&\overset{(3276)}{=} t\cdot\left[\left(\alpha_H^{-1}\omega_H^{-1}\left(S_H^{-1}\left(\psi_H^{-1}(h_2)\right)_1\right)\right.\right.\\
&\quad\left.\cdot\beta_A^{-2}(ac)\right)\left(\beta_H^{-1}\psi_H^{-1}\left(S_H^{-1}\left(\psi_H^{-1}(h_2)\right)_2\right)\right.\\
&\quad\left.\left.\cdot\left(\beta_H^{-2}\omega_H^{-1}(h_1)\cdot\beta_A^{-2}(b)\right)\right)\right]\\
&\overset{(327610)}{=} t\cdot\left[S_H^{-1}\psi_H^{-1}(h_2)\right.\\
&\quad\left.\cdot\left(\beta_A^{-2}(ac)\left(\beta_H^{-2}\omega_H^{-1}(h_1)\cdot\beta_A^{-2}(b)\right)\right)\right]\\
&\overset{(3276107)}{=} \left[tS_H^{-1}\psi_H^{-1}(h_2)\right]\\
&\quad\cdot\left(\beta_A^{-1}(ac)\left(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_A^{-1}(b)\right)\right)\\
&= t\cdot\left(\beta_A^{-1}(ac)\left(\beta_H^{-1}\omega_H^{-1}(h_1)\varepsilon_H(h_2)\cdot\beta_A^{-1}(b)\right)\right)\\
&= t\cdot\left(\beta_A^{-1}(ac)\left(\beta_H^{-1}(h)\cdot\beta_A^{-1}(b)\right)\right)\\
&= t\cdot\beta_A^{-1}((ac)(h\cdot b))\\
&\overset{(32761077)}{=} t\cdot\beta_A^{-1}\left(\alpha_A(a)\left(c\beta_A^{-1}(h\cdot b)\right)\right)\\
&= \left(\alpha_A(a), c\beta_A^{-1}(h\cdot b)\right)\\
&= (\alpha_A(a), (c\#h)\longrightarrow b).
\end{aligned}\tag{37}$$

Finally, the BiHom-associativity is obtained by

$$\begin{aligned}
[a,b]\longrightarrow\beta_A(c) &= [(a\#t)(b\#1_H)]\longrightarrow\beta_A(c)\\
&\overset{(7)}{=}(\alpha_A(a)\#t)\longrightarrow[(b\#1_H)\longrightarrow c]\\
&= (\alpha_A(a)\#t)\longrightarrow\left(b\beta_A^{-1}(1_H\cdot c)\right)\\
&= (\alpha_A(a)\#t)\longrightarrow(bc) = \alpha_A(a)\beta_A^{-1}(t\cdot(bc))\\
&= \alpha_A(a)\left(t\cdot\beta_A^{-1}(bc)\right) = \alpha_A(a)(b,c)\\
\alpha_A(a)\longleftarrow[b,c] = \alpha_A(a)&\longleftarrow[(b\#t)(c\#1_H)]\\
&\overset{(7)}{=}(a\longleftarrow(b\#t))\longleftarrow(\beta_A(c)\#1_H)\\
&= \left[S_H^{-1}\alpha_H^{-1}\beta_H(t)\cdot\beta_A^{-1}(ab)\right]\longleftarrow(\beta_A(c)\#1_H)\\
&= \left(t\cdot\beta_A^{-1}(ab)\right)\longleftarrow(\beta_A(c)\#1_H)\\
&= 1_H\cdot\beta_A^{-1}\left(\left(t\cdot\beta_A^{-1}(ab)\right)\beta_A(c)\right)\\
&= \left(t\cdot\beta_A^{-1}(ab)\right)\beta_A(c) = (a,b)\beta_A(c).
\end{aligned}\tag{38}$$

We get $[A^H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, A\#H]$ forms an associated Morita context.

Data Availability

There is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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